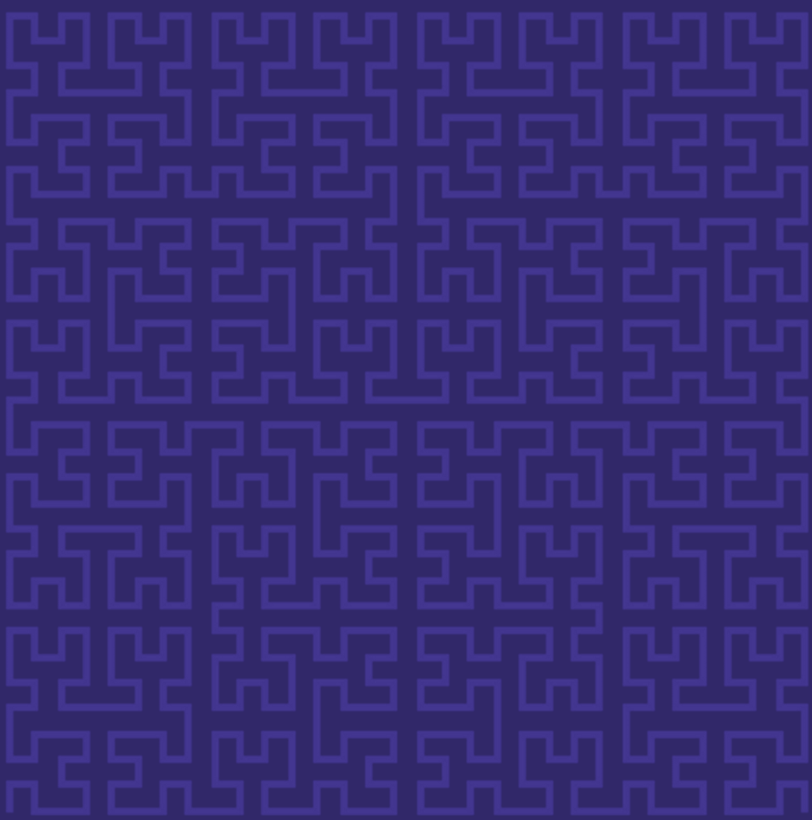
Stability analysis in continuous and discrete time



Niels Besseling

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Paranimfen:

Tim Besseling
Rens Besseling

UNIVERSITEIT TWENTE. Universiteit Twente, Faculteit Elektrotechniek, Wiskunde en Informatica, Vakgroep Wiskundige Systeem- en Besturingstheorie.

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CONTINUOUS AND DISCRETE TIME

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prof. dr. H.J. Zwart
prof. dr. A.A. Stoorvogel

This thesis is dedicated to my parents, Nico & Willy.

Preface

This dissertation is the result of four years of mathematical research I carried out at the department of Applied Mathematics of the University of Twente between 2007 and 2011. Parts of this research have been published in scientific journals. Financial support by the Netherlands Organisation for Scientific Research (NWO) is greatly acknowledged. This dissertation would not have existed without the help of several people, some of whom I would like to mention below.

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Niels Besseling
Enschede, December 2011

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Chapter 1

Introduction

1.1 Motivation

Differential equations provide the basis for mathematical models in many fields in engineering, physics, and economics. They describe the change of the quantities of the physical or economical system with respect to time and place. Together with an initial state, we have an initial-value problem. A general linear initial-value problem is described by the following equation.

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0, \tag{1.1}$$

Starting at $x(0) = x_0$, the change of the system state x is given by operator A.

Equation (1.1) is not restricted to first-order linear equations, since higher order linear systems can be reduced to this first-order one by increasing the dimension of the state space.

If the state space, i.e., the space in which x takes its values, is finite-dimensional, then A will be a matrix. In this case equation (1.1) covers ordinary differential equations.

If the state space is a Hilbert or Banach space, then the operator A can be a differential operator. In this way equation (1.1) covers partial differential equations as well.

The solution of this equation is given via a family of operators $(e^{At})_{t\geq 0}$, which maps the initial state x_0 to the state at time t by

$$x(t) = e^{At}x_0, \qquad t \geq 0. \tag{1.2}$$

This family of operators is called a C_0-*semigroup*. The semigroup maps an initial state to the future states. It has the following defining properties:

$$\begin{aligned} &e^{A(t+s)} = e^{At}e^{As}, \qquad t,s \geq 0, \\ &e^{A0} = I, \\ &\|e^{At}x_0 - x_0\| \to 0, \qquad \text{as } t \to 0^+, \quad \text{for all } x_0 \in X. \end{aligned}$$

Note that in this definition t is assumed to be non-negative. However, for some systems these properties hold for all $t \in \mathbb{R}$. In this case the family of operators is called a *group*.
In general it is not possible to calculate the semigroup corresponding to a generator A. To numerically solve the initial-value problem one could use difference methods. In this method the differential equation (1.1) is replaced by a difference equation.

$$x\big((n+1)\Delta\big) = Qx(n\Delta), \qquad x(0) = x_0, \tag{1.3}$$

where constant Δ is the time step and Q is an approximation of $e^{\Delta A}$, since we can not calculate $e^{\Delta A}$. Time is now denoted by n which we take from $\mathbb{N}$, the set of non-negative integers.
For an approximation of the exponential function, there are several methods. However, explicit methods, like Euler and Runge-Kutta, require impractically small time steps for approximating stiff equations. Since A does not need to be a bounded operator, equation (1.1) can be stiff. Therefore, it is better to look at implicit, A-stable methods like Padé approximations, which do better at a larger time step, see [15].
This means that the difference operator Q is a rational function and the difference equation (1.3) takes the following form.

$$x_{n+1} = r(\Delta A)x_n, \qquad \text{with } x_n = x(n\Delta). \tag{1.4}$$

In the Crank-Nicolson scheme, see [32], r is chosen as the Möbius transform,

$$r(s) = \frac{s/2+1}{-s/2+1}. \tag{1.5}$$

The first three terms of the Taylor series of the Möbius transform $r(s)$ are equal to the first three terms of the Taylor series of e^s, so $r(s) \approx e^s$.
A natural question is whether the solution $r(\Delta A)^n x_0$ of equation (1.4) is a good approximation of the solution $e^{At}x_0$ of (1.1) on a time interval $[0,T]$. Another question is, if the behaviour of the solutions is the same on the long run, as time goes to infinity. We will not discuss the first question, but concentrate on the last one.

The discretisation step Δ is very important in numerical schemes. However, since we are looking to the limit behaviour of abstract semigroups, we may scale the time axis. Thus without loss of generality, we may take $\Delta = 2$.
The operator function corresponding to the Möbius transform (1.5) is known as the *Cayley transform* and was introduced by von Neumann, [27]. For historical reasons the Cayley transform is defined as minus the Möbius transform. We denote it by A_d.

$$A_d = (A + I)(A - I)^{-1}. \tag{1.6}$$

An other important application of the Cayley transform comes from systems theory, see [28] and [9]. We consider the following example of a controlled system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A is the generator of a C_0-semigroup in a Hilbert space X. The operator $A^{-1}B$ is a bounded control operator from the control space U to the state space X. The discrete-time counterpart of this equation can be obtained using the Cayley transform of A, i.e. A_d, and $B_d = \sqrt{2}(I - A)^{-1}B$:

$$x(n + 1) = A_d x(n) + B_d u(n).$$

The Cayley transform maps the unbounded operators A and B of the continuous-time system into bounded operators in the discrete-time counterpart. This certainly brings a technical advantage, and it turns out that control properties, such as controllability, are the same for both systems, see e.g. [9]. The Cayley transform is a link between continuous-time and discrete-time systems. It maps the generator in the continuous-time system A to its *cogenerator* A_d of the corresponding discrete-time system.
As mentioned previously, we are interested the limit behaviour, in particular, the relation in stability between these systems. So do the systems have the same behaviour when $t \to \infty$, and $n \to \infty$, respectively?
The Möbius transform (1.5) maps the closed left half-plane, which is denoted by $\overline{\mathbb{C}}_-$, into the closed interior of the unit disc, which is denoted by $\overline{\mathbb{D}}$. For the eigenvalues of A and A_d this means,

$$\sigma_p(A) \subset \overline{\mathbb{C}}_- \iff \sigma_p(A_d) \subset \overline{\mathbb{D}}. \tag{1.7}$$

Here σ_p denotes the point spectrum, that is the set of all the eigenvalues. So one would expect that the Cayley transform of a stable system is again a stable system. However, in general this is not the case. In Example 1.1 we examine a stable continuous-time system which is mapped by the Cayley transform to a discrete-time system which is not stable.

In this thesis we examine the stability relation between a continuous-time and a corresponding discrete-time system. If we know that the solutions of the differential equation (1.1) are stable, what can be said about the solutions of the difference equation (1.3) and $\|A_d^n\|$? Hence, we want to investigate the relation between the behaviour of the semigroup $(e^{At})_{t\geq 0}$ and the power sequence $(A_d^n)_{n\in\mathbb{N}}$. The precise definitions of stability can be found in Chapter 2.

In the following section we look at the overshoot of a system. In Sections 1.3 and 1.4 we summarize the known results for Banach and Hilbert spaces.

1.2 Overshoot

For most people an exponential function is either a increasing or a decreasing function. Thus by a stable system this function is expected to be decreasing. If $\|e^{At}\| \leq 1$ for all $t \geq 0$, then this holds since

$$\|e^{At}\| = \|e^{A(t-s)}e^{As}\| \leq \|e^{As}\| \leq 1, \qquad \text{for } t > s \geq 0. \tag{1.8}$$

for every $t \geq 0$ the norm is not bigger than one. Semigroups satisfying $\|e^{At}\| \leq 1$ for all $t \geq 0$ are called *contraction semigroups*. However, in general equation (1.8) will not hold. Stable semigroups can have an overshoot. This means that the norm of the semigroup can grow initially, but go to zero afterwards. See for example in Figure 1.1 the plot of a stable system $\|e^{At}\|$. The semigroup is generated by a 7×7-matrix. We can see a huge overshoot in the initial period and the system does not seem to be stable.

This is a stable matrix having all its eigenvalues on the left half plane. With the location of the eigenvalues one can determine the stability properties of the system, but not the transient behaviour. For more details on the matrix and the behaviour of the system we refer to [31]. Nice estimates for bounds on the overshoot are given by Davies in [12]. More information can be found in [38] and [25].

One had to be careful when obtaining numerical solutions of systems with a large overschoot. For this finite-dimensional system the Cayley transform is stable and follows the wild transient behaviour of the continuous time system. In Figure 1.2 we can see this. Here we plotted the powers of the Cayley transform for the time step $\Delta = 0.1$.

However, for infinite-dimensional systems this in not guaranteed. The Cayley transform of a bounded infinite-dimensional system can be unbounded, see Example 1.2.

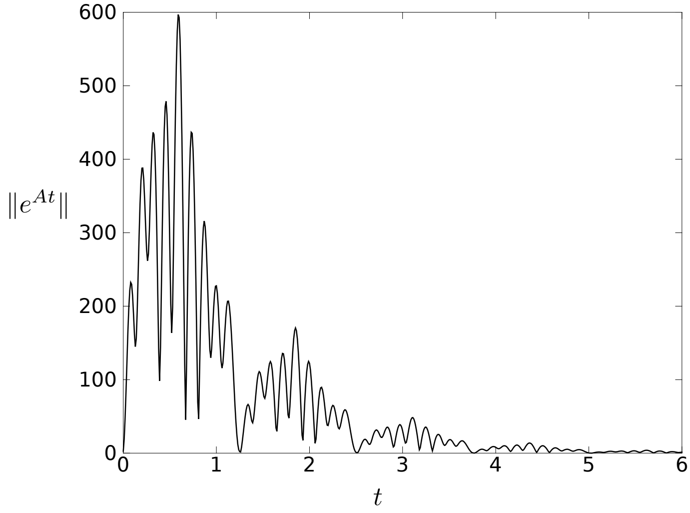

Figure 1.1: Transient behaviour of the stable semigroup.

1.3 Banach space

Van Dorsselaer, Kraaijevanger and Spijker [37] studied the general problem of establishing upper bounds for the power sequence $(A_d^n)_{n\in\mathbb{N}}$ for the case when X is a finite-dimensional Banach space. For these finite-dimensional systems, the Cayley transform does not change the asymptotic stability properties of the solutions, since the eigenvalues are mapped according to equation (1.7).
For example, if for the semigroup there holds:

$$\|e^{At}\| \leq M, \qquad t \geq 0,$$

then its eigenvalues are in $\overline{\mathbb{C}}_-$, that is the closed left half-plane.
and by equation (1.7) the powers of the Cayley transform are bounded, that is, $\|A_d^n\| \leq M_d$ for some M_d. Surprisingly, M_d is also a function of the dimension of the state space. The best bound is given by

$$\|A_d^n\| \leq \min(m, n+1)\, e\, M, \qquad \text{for all } n \in \mathbb{N}, \tag{1.9}$$

where m is the dimension of the space X.
Thus this bound depends on the dimension m of the space X. For infinite-dimensional systems this suggests that the best bound for the powers of the Cayley transform is given by n, i.e. $\|A_d^n\| \leq M_1 n$.

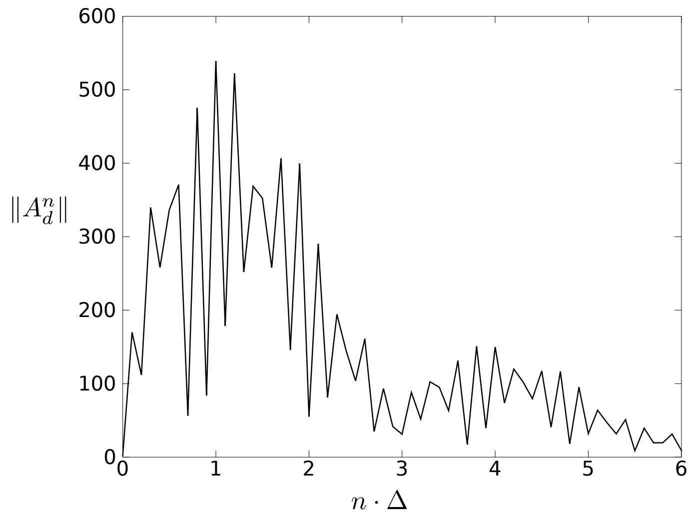

Figure 1.2: Transient behaviour of the Cayley transform.

However, it turns out that the best estimate is given by $M_1\sqrt{n}$, that is $\|A_d^n\| \leq M_1\sqrt{n}$, see Lemma 1.1.
In Example 1.2 we show a stable continuous-time system, for which the powers of Cayley transform grow like $\sqrt{n}$.
In Banach spaces there is the famous result of Brenner and Thomée, [7], giving an upper bound.

Lemma 1.1 *For each rational function r with $|r(z)| \leq 1$ for $\mathrm{Re}(z) \leq 0$ there is a constant M_1 such that for a C_0-semigroup e^{At}, satisfying $\|e^{At}\| \leq M$,*

$$\|r^n(\Delta A)\| \leq MM_1\sqrt{n}, \qquad t = n\Delta.$$

This result holds in particular for the Cayley transform since $\left|\frac{z+1}{z-1}\right| \leq 1$ for $\mathrm{Re}(z) \leq 0$. For the Cayley transform this estimate is sharp, as can be seen in the next example.

Example 1.2 *In this example we show that for a specific contraction semigroup, the powers of the cogenerator grow like $\sqrt{n}$. We provide the idea of the proof without going into technical details.*
Let X be $C_0(0,\infty)$, which is the space of bounded continuous functions on $(0,\infty)$ vanishing at infinity. With the maximum norm this is a Banach space. We consider the left-translation semigroup on X given by

$$(e^{At}f)(s) = f(s+t), \qquad f \in C_0(0,\infty),\ t \geq 0.$$

It is easy to see that this semigroup is a contraction.
By [20, equation 13] we know that we can write the cogenerator in the following way,

$$A_d^n = I - 2\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) e^{At} dt, \qquad n \in \mathbb{N},$$

where $L_{(n-1)}^{(1)}(2t)$ *are the generalized Laguerre polynomials. We treat this relation in more detail in Section 4.4 and Chapter 6.*
Thus,

$$(A_d^n f)(s) = f(s) - 2\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) f(s+t) dt.$$

and

$$\begin{aligned}\|f\| + \|A_d^n f\| &\geq 2\left\|\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) f(s+t) dt\right\| \\ &= \max_{s\geq 0} 2\left|\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) f(s+t) dt\right| \\ &\geq 2\left|\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) f(t) dt\right| \qquad (1.10)\end{aligned}$$

If we let f *approximate the sign-function of* $e^{-t} L_{n-1}^{(1)}(2t)$*, i.e.*

$$f(t) \approx \operatorname{sign}(e^{-t} L_{n-1}^{(1)}(2t)),$$

then

$$\begin{aligned}\|f\| + \|A_d^n f\| &\geq 2\left|\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) f(t) dt\right| \\ &\approx 2\int_0^\infty \left|e^{-t} L_{n-1}^{(1)}(2t)\right| dt \\ &\geq c\sqrt{n}.\end{aligned}$$

In the last step we used an estimate from [1, page 707]. Thus the bound in Lemma 1.1 is sharp.

For other examples see [8], [5], [6].
Example 1.2 shows that for the Cayley transform Lemma 1.1 is sharp in general. However, if we focus on exponentially stable semigroups this result can be improved. The definition of exponentially stability is given in Section 2.1.

Lemma 1.3 *Let A generate an exponentially stable C_0-semigroup, that is $\|e^{At}\| \leq Me^{-\omega t}$, $M, \omega > 0$, on the Banach space X, and let A_d be its cogenerator, then there exists a constant M_1 such that*

$$\|(A_d)^n\| \leq M_1 n^{1/4}, \qquad n \geq 1.$$

This estimate is sharp as well. For the proof we refer to [19].
For bounded semigroups, there are other results. In [18] Gomilko and Zwart present two technical sufficient conditions on the resolvent operator under which the powers of the Cayley transform are bounded. However, these conditions are not so easy to check.
Focussing on more specific Banach spaces, lower growth bounds can be found. Montes-Rodriguez, Sanchez-Alvarez and Zemanek, [26], look at the Volterra operator V defined on $L^p[0,1]$ by

$$(Vf)(x) = \int_0^x f(t)dt, \qquad \text{for } f \in L^p[0,1]. \tag{1.11}$$

For

$$(Af)(x) = f(x) - 2f'(x), \qquad D(A) = \left\{ f \in S_p^1(0,1) \mid f(0) = 0 \right\},$$

where S_p^1 is a Sobolev space, its Cayley transform is given by $A_d = I - V$.

Lemma 1.4 *Let operator V be the Volterra operator given by equation (1.11). For the operator $I - V$ the following relation holds:*

$$\|(I-V)^n\|_p \approx n^{|1/4 - 1/(2p)|}, \qquad \text{for } n \geq 1.$$

In particular, $I - V$ is bounded on $L^p[0,1]$ if and only if $p = 2$.

For the proof we refer to [26, Theorem 1.1].
Lemma 1.4 corresponds to Lemma 1.3 since the growth of $(I - V)^n$ for $n \to \infty$ is bounded by $n^{1/4}$.
Another result for the Volterra operator V on $L^p[0,1]$ is given by Gomilko, see [17]. He shows that for every $p \in [1, \infty]$

$$\overline{\lim_{t \to +\infty}} \left(t^{-|1/4 - 1/(2p)|} \|e^{-tV}\|_{L_p} \right) > 0.$$

With this result we end the section on Banach spaces and focus on Hilbert spaces.

1.4 Hilbert space

In the previous section we examined Banach spaces. From this section on, X will be a Hilbert space. For the growth of the powers of the cogenerator in Hilbert spaces the bound from Lemma 1.1 holds as well. For the Cayley transform however, Gomilko, [20], showed it is possible to get a better estimate.

Lemma 1.5 *Let A generate a bounded C_0-semigroup on Hilbert space X, that is $\|e^{At}\| \leq M$ for all $t \geq 0$, and let A_d be its cogenerator, then there exists a constant M_1 such that*

$$\|(A_d)^n\| \leq M_1 \ln(n+1), \qquad n \in \mathbb{N}.$$

For the proof we refer to [20]. In Chapter 3 we provide an alternative proof for the case that A generates an exponentially stable semigroup.
While for Lemma 1.1 it is known that the bound is sharp, until now this is unknown for the estimate in Lemma 1.5. Moreover, under the conditions of Lemma 1.5 there is no example known for which $\|A_d^n\|$ is unbounded.
However, for some groups of systems stable solutions of the differential equation lead to stable solutions of the difference equation. From [21], [20], and [2], we know the following result

Lemma 1.6 *Let $A \in \mathcal{L}(X)$ be the generator of a bounded semigroup, that is $\|e^{At}\| \leq M$ for all $t \geq 0$, then the power sequence of its cogenerator, $(A_d^n)_{n\in\mathbb{N}}$, is bounded.*

For a proof we refer to Corollary 7.7.
Another well-known case is when A generates a contraction semigroup. A contraction semigroup is a C_0-semigroup $(e^{At})_{t\geq 0}$ with the additional property that $\|e^{At}\| \leq 1$ for $t \geq 0$. Since the norm of a physical system is often directly related to the energy in the system, contraction semigroups correspond to dissipative systems, and so they form a large subclass.
The well-known Lumer-Philips theorem characterizes the generators of contraction semigroups, see [24].

Lemma 1.7 (Lumer-Philips) *A densely defined closed linear operator A generates a contraction semigroup if and only if*

$$\langle Ax, x\rangle + \langle x, Ax\rangle \leq 0, \qquad \text{for all } x \in D(A) \text{ and} \tag{1.12}$$

$$\langle A^*x, x\rangle + \langle x, A^*x\rangle \leq 0, \qquad \text{for all } x \in D(A^*) \tag{1.13}$$

For the proof we refer to [10, Theorem 2.2.2].

A *contraction* is a bounded operator A_d, with the property that the norm $\|A_d\| \leq 1$. Note that the powers of a contraction have the same property, $\|A_d^n\| \leq 1$ for $n \in \mathbb{N}$.
The following lemma characterizes contractions.

Lemma 1.8 *The bounded operator A_d is a contraction if and only if*

$$A_d^* A_d - I \leq 0, \qquad \textit{on } X. \tag{1.14}$$

Proof: The operator A_d is a contraction if and only if

$$\|A_d x\|^2 \leq \|x\|^2, \qquad \text{for all } x \in X.$$

Since we are in a Hilbert space this is equivalent with

$$\langle A_d x, A_d x\rangle - \langle x, x\rangle \leq 0, \qquad \text{for all } x \in X.$$

This is equivalent to inequality (1.14). □

The following theorem states that the Cayley transform maps contraction semigroup to contractions. See [33, Theorem 3.4.9] for a detailed proof, but the result is much older.

Theorem 1.9 *Let A generate a C_0-semigroup and let A_d be its cogenerator. Then A generates a contraction semigroup if and only if A_d is a contraction.*

Proof: Assume that A_d is a contraction. By Lemma 1.8 and relation (1.14) this is equivalent to

$$\langle A_d x, A_d x\rangle - \langle x, x\rangle \leq 0, \qquad \text{for all } x \in X.$$

This is equivalent to

$$\langle (A+I)(A-I)^{-1}x, (A+I)(A-I)^{-1}x\rangle - \langle x, x\rangle \leq 0, \quad \text{for all } x \in X. \tag{1.15}$$

For all $y \in D(A)$ there exists a $x \in X$ such that $x = (A-I)y$. Thus we can reformulate relation (1.15) as follows.

$$\langle (A+I)y, (A+I)y\rangle - \langle (A-I)y, (A-I)y\rangle \leq 0, \qquad \text{for all } y \in D(A),$$

which implies inequality (1.12).
Relation (1.13) is proved similarly using $\|A_d\| = \|A_d^*\|$. Lemma 1.7 implies that A generates a contraction semigroup.

To prove the other implication, we assume that A generates a contraction semigroup. Using Lemma 1.7 and relation (1.12), we know that

$$\langle (A+I)y, (A+I)y\rangle - \langle (A-I)y, (A-I)y\rangle \leq 0, \qquad \text{for all } y \in D(A). \tag{1.16}$$

The operator $A - I$ maps $D(A)$ onto X and so for all $x \in X$ there exists a $y \in D(A)$ such that $y = (A-I)^{-1}x$. We can write relation (1.16) as follows.

$$\langle A_d x, A_d x\rangle - \langle x, x\rangle \leq 0, \qquad \text{for all } x \in X,$$

which implies inequality (1.14).
Lemma 1.8 implies that A_d is a contraction and this completes the proof. □

Another class of operators for which the relation between the boundedness of the semigroup and the boundedness of the powers of the cogenerator is known, is the class of analytic semigroups.
First we define for $\alpha \in (0, \frac{\pi}{2})$ the sector Δ_α in the complex plane by

$$\Delta_\alpha := \{t \in \mathbb{C} \mid |\arg(t)| < \alpha, t \neq 0\}.$$

A C_0-semigroup $(e^{At})_{t\geq 0}$ is *analytic* if it can be continued analytically into Δ_α for an $\alpha > 0$.

Lemma 1.10 *Operator A generates the analytic semigroup e^{At} and $\|e^{At}\| \leq M$ for all $t \in \Delta_\alpha$ if and only if there exists an $m \geq 0$ such that*

$$\|(sI - A)^{-1}\| \leq \frac{m}{|s|}, \qquad \textit{for all complex } s \textit{ with } |\arg(s)| < \frac{\pi}{2} + \alpha.$$

For the proof we refer to Pazy, [30, Theorem 2.5.2].

Theorem 1.11 *Let A generate an analytic C_0-semigroup on the Hilbert space X and let A_d be its cogenerator. If $(e^{At})_{t\geq 0}$ is bounded, then the powers of the cogenerator $(A_d^n)_{n\in\mathbb{N}}$ are bounded as well.*

For the proof we refer to [21, Theorem 6.1]. See also Chapter 7
There is a relation from the discrete-time system to the continuous-time system as well. This relation involves bounded and strongly stable power sequences. In Section 2.1 and Section 2.2 we give a definitions of these properties.

Theorem 1.12 *Let A_d be the Cayley transform of operator A and the powers $(A_d^n)_{n\in\mathbb{N}}$ are bounded. If -1 is not an eigenvalue of A_d and A_d^*, the adjoint of A_d, and if there exist a $C > 0$ and $\delta > 0$ such that*

$$|\mu + 1|\|R(\mu, A_d)\| \leq C, \qquad \textit{for all } |\mu + 1| \leq \delta, \mathrm{Re}(\mu) < -1.$$

then A is generates a bounded semigroup, which is analytic.
Moreover, if $(A_d^n)_{n\in\mathbb{N}}$ is strongly stable, so is $(e^{At})_{t\geq 0}$.

For the proof we refer to [21, Theorem 7.1].

1.5 General lemmas

For future reference we list here simple relations between the operator A and its cogenerator A_d. We start with the definition of the cogenerator A_d, which is the Cayley transform of operator A.
Let A be a densely defined closed operator and let $1 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. The cogenerator A_d is defined as

$$A_d = (A + I)(A - I)^{-1}.$$

By simple manipulations we obtain another way to write the Cayley transform.

Lemma 1.13 *Let A_d be the Cayley transform of A. Then another way to write A_d is given by,*

$$A_d = I + 2(A - I)^{-1}. \tag{1.17}$$

Proof: Equation (1.17) is easily derived from the definition of A_d.

$$\begin{aligned} A_d &= (A + I)(A - I)^{-1} \\ &= (A - I + 2I)(A - I)^{-1} \\ &= I + 2(A - I)^{-1}. \end{aligned}$$

This proves the lemma. □

The inverse of the Cayley transform is the Cayley transform itself. This implies that the Cayley transform of the cogenerator A_d is the operator A.

Lemma 1.14 *Let A_d be the Cayley transform of A. Then A is given by,*

$$A = (A_d + I)(A_d - I)^{-1}, \qquad D(A) = \operatorname{ran}(A_d - I). \tag{1.18}$$

Proof: From Lemma 1.13 we know that $A_d - I = 2(A - I)^{-1}$ so the inverse operator $(A_d - I)^{-1}$ with domain $\operatorname{ran}(A_d - I)$ exist.
On its domain $\operatorname{ran}(A_d - I)$ we can substitute equation (1.17) in the right-hand side of equation (1.18).

$$\begin{aligned} (A_d + I)(A_d - I)^{-1} &= \left[2I + 2(A - I)^{-1}\right]\left[2(A - I)^{-1}\right]^{-1} \\ &= \left[I + (A - I)^{-1}\right](A - I) \\ &= A - I + I = A. \end{aligned}$$

So $D(A) = \operatorname{ran}(A_d - I)$ and equation (1.18) holds.
This proves the lemma. □

After the inverse of the Cayley transform, we look at the Cayley transform of the inverse.

Lemma 1.15 *Let A be a densely defined closed linear operator on X and assume that the inverse A^{-1} exists.*
For the Cayley transform of A^{-1} the following relation holds:

$$(A^{-1})_d = -A_d, \qquad \text{on } X.$$

Proof: From $1 \in \rho(A)$ follows that $1 \in \rho(A^{-1})$.
On X the following relations holds:

$$\begin{aligned}(A^{-1})_d &= (A^{-1}+I)(A^{-1}-I)^{-1} = (I+A)(I-A)^{-1} \\ &= -A_d.\end{aligned}$$

This proves the lemma. □

In the following lemma, we define the operator A_s and derive its Cayley transform.

Lemma 1.16 *Let A be a densely defined closed linear operator on X with its spectrum contained in the left half-plane, i.e. $\sigma(A) \subset \{\lambda \in \mathbb{C} | \mathrm{Re}(\lambda) \le 0\}$. Let A_d be the cogenerator of A.*
If for some $s \in \mathbb{R}$ the inverse of $A - isI$ exists as a closed operator, then the operator A_s defined as

$$A_s = (-isA + I)(A - isI)^{-1}, \qquad D(A_s) = \mathrm{ran}(A - isI), \tag{1.19}$$

is a densely defined closed operator as well. Furthermore,

$$A_s = -isI + (s^2+1)(A - isI)^{-1}, \tag{1.20}$$

and

$$(A_s)_d = \alpha(s)A_d, \qquad \text{where } \alpha(s) = (is-1)(is+1)^{-1}. \tag{1.21}$$

Note that $|\alpha(s)| = 1$, and hence the growth of A_d and $(A_s)_d$ are the same.

Proof: From the assumptions on A and the definition of the operator A_s, it follows that A_s is a densely defined closed linear operator.

$$A_s = (-isA - s^2 I + s^2 I + I)(A - isI)^{-1} \tag{1.22}$$
$$= -isI + (s^2+1)(A - isI)^{-1}, \tag{1.23}$$

which proves equation (1.20).
Using the equality

$$A_s - I = -(is+1)(A - I)(A - isI)^{-1},$$

we find that

$$\begin{aligned}(A_s - I)^{-1} &= -(is+1)^{-1}(A - isI)(A-I)^{-1}\\ &= -(is+1)^{-1}(A - I - (is-1)I)(A-I)^{-1}\\ &= -(is+1)^{-1}I + \alpha(s)(A-I)^{-1}\\ &= \frac{(is-1)-(is+1)}{2}(is+1)^{-1}I + \alpha(s)(A-I)^{-1}\\ &= \frac{\alpha(s))-1}{2}I + \alpha(s)(A-I)^{-1}.\end{aligned}$$

Thus

$$(A_s)_d = I + 2(A_s - I)^{-1} = \alpha(s)I + 2\alpha(s)(A-I)^{-1} = \alpha(s)A_d,$$

which proves the assertion. □

With $s = 0$ we see that the growth of A_d and $(A^{-1})_d$ are identical.
The next remark is a consequence of Lemma 1.16.

Remark 1.17 *From equation (1.20) we conclude that $(A - isI)^{-1}$ is the generator of a C_0-semigroup if and only if the operator A_s is the generator of a C_0-semigroup. Moreover,*

$$\|e^{A_s t}\| = \|e^{(s^2+1)(A-isI)^{-1}t}\|, \qquad t \geq 0.$$

1.6 Overview

In this thesis we look at the relation between continuous-time systems and their corresponding discrete-time systems via the Cayley transform. We examine the question how the growth of the powers of the cogenerator is related to the growth of the semigroup.
In particular, we examine the stability relations. In Chapter 2 we give the definitions of stability and show some important results which we need in the other chapters.
In Chapter 3 we prove that for exponentially stable semigroups on a Hilbert space the growth of the powers of their cogenerator is bounded by $\ln(n+1)$. Although this result was shown before by Gomilko, we provide a different proof using Lyapunov equations. Chapter 3 has been published as an internal report, [4].
In Chapter 4 we define the notion of Bergman distance as a distance between semigroups as well as a distance between cogenerators. If two semigroups have a finite Bergman distance, they have the same stability properties.

For cogenerators the same holds. Furthermore, the Bergman distance is preserved by the Cayley transform. With this notion we are able to extend the class of stable semigroups with stable cogenerators. Chapter 4 is based on [3].
Equivalence classes defined by the Bergman distance are examined in Chapter 5. We look at some class properties and give a characterization of equivalence classes in finite-dimensional state spaces.
In Chapter 6 we examine and extend the proof from Section 4.4. There we proved the preservation of the Bergman distance by the Cayley transform using Laguerre polynomials. By including more general Laguerre polynomials we can extend this method.
In Chapter 7 we introduce the inverse of A to get further stability results. We provide sufficient conditions under which the growth bounds on semigroup generated by the inverse hold as well for the powers of the cogenerator, and the other way around. Furthermore, if A and A^{-1} generate a bounded semigroup, the powers of the cogenerator is bounded as well. Chapter 7 is based on [19].

Chapter 2

Stability

In this chapter we summarize some stability results for continuous and discrete time systems. The state space in this section is the Hilbert space X. First we have a quick look at the systems already introduced in Chapter 1. The continuous time system is given by the abstract differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{2.1}$$

We assume that A is the infinitesimal generator of the C_0-semigroup $(e^{At})_{t\geq 0}$. Thus the (weak) solution of (2.1) is given by

$$x(t) = e^{At}x_0, \qquad t \geq 0. \tag{2.2}$$

Our discrete time system is given by the difference equation,

$$x_d(n+1) = A_d x_d(n), \qquad x_d(0) = x_0, \tag{2.3}$$

with $A_d \in \mathcal{L}(X)$. The space $\mathcal{L}(X)$ is the space of bounded linear operators on X.
The solution of (2.3) is given by,

$$x_d(n) = A_d^n x_0, \qquad n \in \mathbb{N}. \tag{2.4}$$

We begin with the study of stable continuous-time systems in Section 2.1. In Section 2.2 we study stability of discrete-time systems. Preliminary results on the relation between stability of the C_0-semigroup and the powers of the difference operator are given in section 2.3.

2.1 Continuous-time case

For the system (2.1) we distinguish several kinds of stability, which are defined next.

Definition 2.1 *The C_0-semigroup $(e^{At})_{t\geq 0}$ is* bounded *if there exists a constant $M \geq 1$ such that*

$$\|e^{At}\| \leq M, \qquad \text{for all } t \geq 0. \tag{2.5}$$

The C_0-semigroup $(e^{At})_{t\geq 0}$ is exponentially stable *if there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$\|e^{At}\| \leq Me^{-\omega t}, \qquad \text{for all } t \geq 0. \tag{2.6}$$

Exponentially stable semigroups have a growth bound *which is defined by*

$$\omega_0 = \inf\left\{\omega \in \mathbb{R} \mid \exists M_\omega \geq 1 \text{ such that } \|e^{At}\| \leq M_\omega e^{\omega t}\, \forall\, t \geq 0\right\} \tag{2.7}$$

The C_0-semigroup $(e^{At})_{t\geq 0}$ is strongly stable *if for all $x_0 \in X$,*

$$e^{At}x_0 \to 0, \qquad \text{as } t \to \infty. \tag{2.8}$$

So the difference between strongly stable and exponentially stable is that in the second case the solutions of equation (2.1) converge to zero exponentially.
The smallest constant M for which equation (2.5) holds, is called the *overshoot* of the semigroup.
We have defined stability as a property of the C_0-semigroup. However, since the semigroup is directly linked to the abstract differential equation (2.1), we will sometime say that the equation (2.1) is bounded, exponentially stable or strongly stable.
For the adjoint semigroup $(e^{A^*t})_{t\geq 0}$, the following holds.

Remark 2.2 *The following properties hold for the adjoint semigroup:*

- *$(e^{At})_{t\geq 0}$ is bounded if and only if $(e^{A^*t})_{t\geq 0}$ is bounded,*
- *$(e^{At})_{t\geq 0}$ is exponentially stable if and only if $(e^{A^*t})_{t\geq 0}$ is exponentially stable.*

For strong stability such an equivalence does not hold. For example, the left translation semigroup on $X = L^2(0,\infty)$, given by

$$(e^{At}f)(s) = f(s+t), \qquad f \in L^2(0,\infty), \quad s,t \geq 0, \tag{2.9}$$

is strongly stable. However its adjoint, given by

$$(e^{A^*t}f)(s) = \begin{cases} f(s-t) & s \geq t, \\ 0 & s < t, \end{cases} \tag{2.10}$$

is not strongly stable, since $\|e^{A^*t}f\| = \|f\|$, *for all* $t \geq 0$.

Since stability plays an important role in all kind of applications, it is important to have (simple) characterizations of it.
For exponential stability we have the following sufficient condition:

Lemma 2.3 *Let A generate a C_0-semigroup. There exists a time t_0 such that*

$$\|e^{At_0}\| = r < 1, \tag{2.11}$$

if and only if the semigroup $(e^{At})_{t\geq 0}$ is exponentially stable.

Proof: Let time t_0 satisfy equation (2.11). Define

$$\omega = \frac{-\ln r}{t_0}, \qquad \text{and} \qquad M = \sup_{t_1 \in (0,t_0)} \|e^{(A+\omega)t_1}\|.$$

Now we have $\omega > 0$,

$$\|e^{At_0}\| = e^{-\omega t_0}, \qquad \text{and} \qquad \|e^{At_1}\| \leq Me^{-\omega t_1}, \text{ for all } t_1 \in [0, t_0).$$

For every $t \geq 0$ we can write $t = nt_0 + t_2$, with $n \in \mathbb{N}$ and $t_2 \in [0, t_0)$. Using the semigroup properties and the estimates above we get,

$$\begin{aligned} \|e^{At}\| &= \| \left(e^{At_0}\right)^n e^{At_2}\| \\ &\leq e^{-n\omega t_0} Me^{-\omega t_2} = Me^{-\omega t}. \end{aligned}$$

Hence the semigroup $(e^{At})_{t\geq 0}$ is exponentially stable.
If $(e^{At})_{t\geq 0}$ is exponentially stable and $\|e^{At}\| \leq Me^{-\omega t}$, then all $t_0 > \frac{\ln M}{\omega}$ satisfy equation (2.11). □

A well-known technique for proving stability of a general, possible non-linear, system is the construction of a Lyapunov function. For the linear system (2.1) this technique can be used to characterize exponential stability. The following lemma by Datko is important for the proof of this equivalence.

Lemma 2.4 (Datko) *Let A be the infinitesimal generator of the C_0-semigroup $(e^{At})_{t\geq 0}$ on the Hilbert space X. Then $(e^{At})_{t\geq 0}$ is exponentially stable if and only if*

$$\int_0^\infty \|e^{At}x_0\|^2 dt < \infty, \tag{2.12}$$

for all $x_0 \in X$.

For the proof we refer to Datko [11] or Curtain and Zwart [10, Lemma 5.1.2].

Theorem 2.5 *Let A generate a C_0-semigroup on the Hilbert space X. Then $(e^{At})_{t\geq 0}$ is exponentially stable if and only if there exists a positive operator $Q \in \mathcal{L}(X)$ such that*

$$A^*Q + QA = -I, \qquad \text{on } D(A). \tag{2.13}$$

For the proof we refer to Curtain and Zwart [10, Lemma 5.1.3]. Equation (2.13) is called a *Lyapunov equation.*

Remark 2.6 *Another way of writing the Lyapunov equation is using its weak form, i.e.,*

$$\langle Ax, Qx\rangle + \langle Qx, Ax\rangle = -\langle x, x\rangle, \qquad \text{for all } x \in D(A). \tag{2.14}$$

It is not hard to show that if equation (2.14) holds, then Q maps $D(A)$ into $D(A^*)$ and so equation (2.14) can be written as equation (2.13). To shorten our formulas we use the form of equation (2.13).

The associated *Lyapunov function* $V(x)$ is given by

$$V(x) = \langle Qx, x\rangle. \tag{2.15}$$

Since Q is positive we have that $V(x) \geq 0$. Furthermore, a small calculation shows that $\dot{V}(x(t))$ is negative.

$$\begin{aligned} \dot{V}(e^{At}x_0) &= \langle QAe^{At}x_0, e^{At}x_0\rangle + \langle Qe^{At}x_0, Ae^{At}x_0\rangle \\ &= -\|e^{At}x_0\|^2, \end{aligned}$$

where we used that Q is positive and equation (2.14). Hence V satisfies the standard properties of a Lyapunov function. This Lyapunov function can be used to verify stability.

Remark 2.7 *If the Lyapunov equation (2.13) has a positive or a non-negative solution, then this solution is unique.*

Remark 2.8 *If the semigroup is exponentially stable, then the solution Q of equation (2.13) is given by*

$$\langle Qx, y\rangle = \int_0^\infty \langle e^{At}x, e^{At}y\rangle dt. \tag{2.16}$$

Hence by using equation (2.6), we have the following estimate for the norm of Q,

$$\|Q\| \leq \frac{M^2}{2\omega}. \tag{2.17}$$

For further details on Lyapunov equations we refer to [10, Chapter 4].
Van Casteren, [36], gave a characterization of bounded semigroups. For completeness we include the proof of this characterization.

Lemma 2.9 *The semigroup* $(e^{At})_{t\geq 0}$ *is bounded if and only if there exists a* M_2 *such that for all* $t \geq 0$*, and all* $x_0 \in X$*,*

$$\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds \leq M_2\|x_0\|^2 \text{ and } \frac{1}{t}\int_0^t \|e^{A^*s}x_0\|^2 ds \leq M_2\|x_0\|^2 \quad (2.18)$$

with M_2 *independent of* t *and* x_0*.*

Proof: Assuming the boundedness of $(e^{At})_{t\geq 0}$, we directly prove the relations in equation (2.18). For $(e^{At})_{t\geq 0}$ we find a M_1 such that $\|e^{At}\| \leq M_1$ for all $t \geq 0$.

$$\int_0^t \|e^{As}x_0\|^2 ds \leq \int_0^t \sup_s \|e^{As}x_0\|^2 ds \leq \int_0^t M_1^2\|x_0\|^2 ds = M_1^2 t\|x_0\|^2.$$

The second inequality in (2.18) is proved similarly.
To prove the converse implication we use the Datko trick in the first three steps.

$$\begin{aligned}
t\|e^{At}\| &= \sup_{\|x\|,\|y\|\leq 1} t \cdot \langle y, e^{At}x\rangle \\
&= \sup_{\|x\|,\|y\|\leq 1} \int_0^t \langle y, e^{At}x\rangle ds \\
&= \sup_{\|x\|,\|y\|\leq 1} \int_0^t \langle e^{A^*s}y, e^{A(t-s)}x\rangle ds \\
&= \sup_{\|x\|,\|y\|\leq 1} \langle e^{A^*\cdot}y, e^{A(t-\cdot)}x\rangle_{L^2((0,t),X)} \\
&\leq \sup_{\|x\|,\|y\|\leq 1} \left(\int_0^t \|e^{A^*s}y\|^2 ds\right)^{\frac{1}{2}} \cdot \left(\int_0^t \|e^{A(t-s)}x\|^2 ds\right)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|,\|y\|\leq 1} \sqrt{M_2 t}\|y\|\sqrt{M_2 t}\|x\| = M_2 t,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the second last inequality. Thus the semigroup is bounded. □

Using the above result, we can characterize strong stability. Note that we look at element-wise behaviour.

Lemma 2.10 *Assume that the semigroup* $(e^{At})_{t\geq 0}$ *is bounded and let* $x_0 \in X$*. Then*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds = 0 \quad \text{if and only if} \quad \lim_{t\to\infty} e^{At}x_0 = 0.$$

Proof: To prove the "$\Rightarrow$" implication, we choose an $\varepsilon > 0$. Since the semigroup is bounded, there exists a M such that $\|e^{At}\| \leq M$ for all $t \geq 0$. The first step is to show that the condition

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds = 0, \tag{2.19}$$

implies that there exists a τ_ε such that

$$\|e^{A\tau_\varepsilon}x_0\| < \frac{\varepsilon}{M}. \tag{2.20}$$

If there does not exist such a τ_ε, then

$$\|e^{At}x_0\| \geq \frac{\varepsilon}{M}, \qquad \text{for all } t \geq 0.$$

Thus

$$\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds \geq \frac{1}{t}\int_0^t \frac{\varepsilon^2}{M^2} ds = \frac{\varepsilon^2}{M^2}, \qquad \text{for all } t \geq 0.$$

This contradicts condition (2.19). Hence, there exists a τ_ε satisfying condition (2.20).
Thus, for all $t > \tau_\varepsilon$,

$$\|e^{At}x_0\| = \|e^{A(t-\tau_\varepsilon)}e^{A\tau_\varepsilon}x_0\| < M\frac{\varepsilon}{M} = \varepsilon.$$

This holds for all $\varepsilon > 0$, so $\lim_{t\to\infty} e^{At}x_0 = 0$.
To prove the "$\Leftarrow$" implication, we choose an $\varepsilon > 0$. Since $\lim_{t\to\infty} e^{At}x_0 = 0$, there exists a τ_ε such that for all $t > \tau_\varepsilon$, $\|e^{At}x_0\| < \varepsilon$. This implies that

$$\begin{aligned}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds &= \frac{1}{t}\int_0^{\tau_\varepsilon} \|e^{As}x_0\|^2 ds + \frac{1}{t}\int_{\tau_\varepsilon}^t \|e^{As}x_0\|^2 ds \\ &\leq \frac{M^2\|x_0\|^2\tau_\varepsilon}{t} + \varepsilon\frac{t-\tau_\varepsilon}{t}\end{aligned}$$

For t large enough, the left-hand side is smaller than 2ε. This holds for all $\varepsilon > 0$, so the left-hand side goes to 0, as t goes to ∞. □

2.2 Discrete-time case

Next, we define what we mean by stability in discrete time.

Definition 2.11 *The operator sequence $(A_d^n)_{n\geq 0}$ is bounded if there exists a constant $M \geq 1$ such that*

$$\|A_d^n\| \leq M, \qquad \text{for all } n \geq 0.$$

The operator sequence $(A_d^n)_{n\geq 0}$ is power stable if there exist constants $M \geq 1$ and $r \in (0,1)$ such that

$$\|A_d^n\| \leq Mr^n, \qquad \text{for all } n \geq 0. \tag{2.21}$$

The operator sequence $(A_d^n)_{n\geq 0}$ is strongly stable if for all $x_0 \in X$,

$$A_d^n x_0 \to 0, \quad \text{as } n \to \infty.$$

So the difference between strongly stable and exponentially stable is that in the second case the solutions of equation (2.3) converge to zero exponentially.
We have defined stability as a property of the operator sequence $(A_d^n)_{n\geq 0}$. However, since the sequence is directly linked to the abstract difference equation (2.3), we will sometime say that the equation (2.3) is bounded, power stable or strongly stable.
For the power sequence of the adjoint difference operator $(A_d^{*n})_{n\geq 0}$, the following holds.

Remark 2.12 *The following properties hold for the adjoint difference operator:*

- *$(A_d^n)_{n\geq 0}$ is bounded if and only if $(A_d^{*n})_{n\geq 0}$ is bounded,*
- *$(A_d^n)_{n\geq 0}$ is power stable if and only if $(A_d^{*n})_{n\geq 0}$ is power stable.*

For strong stability such an equality does not hold. For example, the operator which applies a left translation of size Δ on $X = L^2(0,\infty)$, given by

$$(A_d f)(s) = f(s+\Delta), \qquad f \in L^2(0,\infty), \quad s \geq 0 \tag{2.22}$$

is strongly stable. However its adjoint, given by

$$(A_d^* f)(s) = \begin{cases} f(s-\Delta) & s \geq \Delta, \\ 0 & s < \Delta, \end{cases} \tag{2.23}$$

is not strongly stable, since $\|A_d^* f\| = \|f\|$.

Since stability plays an important role in all kind of applications, it is important to have (simple) characterizations of it.

Lemma 2.13 *If for the operator sequence* $(A_d^n)_{n\geq 0}$ *there exists a* $n_0 \in \mathbb{N}$ *such that*

$$\|A_d^{n_0}\| = R < 1,$$

then the operator sequence $(A_d^n)_{n\geq 0}$ *is power stable.*

Proof: Define

$$r = \sqrt[n_0]{R}, \qquad \text{and} \qquad M = \sup_{0\leq n_1 < n_0} \left\| \left(\frac{A_d}{r} \right)^{n_1} \right\|.$$

Now we have $r < 1$,

$$\|A_d^{n_0}\| = r^{n_0}, \qquad \text{and} \qquad \|A_d^{n_1}\| \leq M r^{n_1}, \text{ for all } 0 \leq n_1 < n_0.$$

For every $n \geq 0$ we can write $n = kn_0 + n_2$, with $k \in \mathbb{N}$ and $0 \leq n_2 < n_0$. This gives the following estimate,

$$\|A_d^n\| = \|\left(A_d^{n_0}\right)^k A_d^{n_2}\| \leq r^{kn_0} M r^{n_2} = M r^n.$$

Hence the operator sequence $(A_d^n)_{n\geq 0}$ is power stable. □

A well-known technique for proving stability of a general, possible non-linear system is the construction of a Lyapunov function. For the linear system (2.3) this technique can be used to characterize power stability. The following lemma is important for the proof of this equivalence.

Lemma 2.14 (Datko) *Let* A_d *be a bounded operator on the Hilbert space* X*. Then* $(A_d^n)_{n\in\mathbb{N}}$ *is power stable if and only if*

$$\sum_{n=0}^{\infty} \|A_d^n x_0\|^2 < \infty, \tag{2.24}$$

for all $x_0 \in X$*.*

Theorem 2.15 *Let* A_d *be a bounded operator on the Hilbert space* X*. Then* $(A_d^n)_{n\in\mathbb{N}}$ *is power stable if and only if there exists a positive operator* $Q \in \mathcal{L}(X)$ *such that*

$$A_d^* Q A_d - Q = -I, \qquad \text{on } X. \tag{2.25}$$

Equation (2.25) is called a *discrete Lyapunov equation.*

Remark 2.16 *Another way of writing the discrete Lyapunov equation is using its weak form, i.e.,*

$$\langle QA_d x, A_d x\rangle - \langle Qx, x\rangle = -\langle x, x\rangle, \qquad \text{for all } x \in X. \tag{2.26}$$

To shorten our formulas we use the form of equation (2.25).

The associated Lyapunov function $V(x)$ is given by

$$V(x) = \langle Qx, x\rangle. \tag{2.27}$$

This is positive since Q is positive. Furthermore, a small calculation shows that $V(x(n+1)) - V(x(n))$ is negative.

$$\begin{aligned} V(A_d^{n+1}x_0) - V(A_d^n x_0) &= \langle A_d^* Q A_d A_d^n x_0, A_d^n x_0\rangle - \langle QA_d^n x_0, A_d^n x_0\rangle \\ &= -\|A_d^n x_0\|^2, \end{aligned}$$

where we used that Q is positive and equation (2.26).

Remark 2.17 *If the discrete Lyapunov equation has a positive or non-negative solution, then this solution is unique.*

Remark 2.18 *If the power sequence $(A_d^n)_{n\in\mathbb{N}}$ is power stable, then Q is given by*

$$\langle Qx, y\rangle = \sum_{n=0}^{\infty} \langle A_d^n x, A_d^n y\rangle. \tag{2.28}$$

Hence by equation (2.21), we have the following estimate for the norm of Q,

$$\|Q\| \leq \frac{M^2}{1 - r^2}. \tag{2.29}$$

So we have linked discrete Lyapunov equation to stability. However, looking more carefully, this connection goes via a quadratic sum. Other quadratic sums can also be related to Lyapunov equations.

Lemma 2.19 *Let A_d be a bounded operator on the Hilbert space X, and let C be a bounded operator on X. Then, for all $x_0 \in X$*

$$\sum_{n=0}^{\infty} \|CA_d^n x_0\|^2 dt < \infty, \tag{2.30}$$

if and only if there exists a non-negative operator Q such that

$$A_d^* Q A_d - Q = -C^* C, \qquad \text{on } X. \tag{2.31}$$

For all Q satisfying equation (2.31) there holds

$$\sum_{n=0}^{\infty} \|CA_d^n x_0\|^2 \leq \langle Qx_0, x_0 \rangle, \tag{2.32}$$

for all $x_0 \in X$.

For the proof we refer to Guo and Zwart [21, Lemma 3.1].

Remark 2.20 *Without additional assumptions we will not have equality in inequality (2.32). For example $Q = I$, $A_d^* = A_d^{-1}$, and $C = 0$ satisfies relations (2.31) but does not have an equality in the last relation.*

The characterization of Van Casteren from the previous section has a discrete time version as well. This version characterizes bounded and strongly stable operator sequences.

Lemma 2.21 (Van Casteren) *The operator sequence $(A_d^n)_{n\geq 0}$ is bounded if and only if there exists a M such that for all $N \in \mathbb{N}$, and all $x_0 \in X$,*

$$\frac{1}{N}\sum_{k=1}^{N} \|A_d^k x_0\|^2 \leq M\|x_0\|^2 \text{ and } \frac{1}{N}\sum_{k=1}^{N} \|A_d^{*k} x_0\|^2 \leq M\|x_0\|^2. \tag{2.33}$$

with M independent of N and x_0.

Proof: Assuming the boundedness of $(A_d^n)_{n\geq 0}$, we directly prove the relations in equation (2.33). For $(A_d^n)_{n\geq 0}$ we find a M_1 such that $\|A_d^n\| \leq M_1$ for all $n \geq 0$.

$$\sum_{k=1}^{N} \|A_d^k x_0\|^2 \leq \sum_{k=1}^{N} \sup_k \|A_d^k x_0\|^2 \leq \sum_{k=1}^{N} M_1^2 \|x_0\|^2 = M_1^2 N \|x_0\|^2.$$

The second inequality in (2.33) is proved similarly.

To prove the converse implication we use the discrete Datko trick in the first three steps.

$$\begin{aligned}
N\|A_d^N\| &= \sup_{\|x\|,\|y\|\leq 1} N\cdot\langle y, A_d^N x\rangle \\
&= \sup_{\|x\|,\|y\|\leq 1} \sum_{k=1}^{N}\langle y, A_d^N x\rangle \\
&= \sup_{\|x\|,\|y\|\leq 1} \sum_{k=1}^{N}\langle A_d^{*k} y, A_d^{N-k} x\rangle \\
&= \sup_{\|x\|,\|y\|\leq 1} \langle A_d^{*\cdot} y, A_d^{N-\cdot} x\rangle_{\ell^2(X)} \\
&\leq \sup_{\|x\|,\|y\|\leq 1} \left(\sum_{k=1}^{N}\|A_d^{*k}y\|^2 ds\right)^{\frac{1}{2}}\cdot\left(\sum_{k=1}^{N}\|A_d^{N-k}x\|^2 ds\right)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|,\|y\|\leq 1} \sqrt{MN}\|y\|\sqrt{MN}\|x\| = MN,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the second last inequality. Thus the operator sequence is bounded. □

Using the above result, we can characterize strong stability. Note that we look at element-wise behaviour.

Lemma 2.22 *Assume that the operator sequence $(A_d^n)_{n\geq 0}$ is bounded and let $x_0 \in X$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\|A_d^k x_0\|^2 = 0 \quad \textit{if and only if} \quad \lim_{N\to\infty} A_d^N x_0 = 0.$$

Proof: To prove the "⇒" implication, we choose an $\varepsilon > 0$. Since the operator sequence is bounded, there exists a M such that $\|A_d^N\| \leq M$ for all $N \geq 0$.
The first step is to show that the condition

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\|A_d^k x_0\|^2 = 0, \tag{2.34}$$

implies that there exists a n_ε such that

$$\|A_d^{n_\varepsilon} x_0\| < \frac{\varepsilon}{M}. \tag{2.35}$$

If there does not exist such a n_ε, then

$$\|A_d^k x_0\| \geq \frac{\varepsilon}{M}, \qquad \text{for all } k \geq 0.$$

Thus

$$\frac{1}{N}\sum_{k=1}^{N}\|A_d^k x_0\|^2 \geq \frac{1}{N}\sum_{k=1}^{N}\frac{\varepsilon^2}{M^2} = \frac{\varepsilon^2}{M^2}, \qquad \text{for all } N \geq 0.$$

This contradicts condition (2.34). Hence, there exists a n_ε satisfying condition (2.35).
Thus, for all $N > n_\varepsilon$,

$$\|A_d^N x_0\| = \|A_d^{N-n_\varepsilon} A_d^{n_\varepsilon} x_0\| < M\frac{\varepsilon}{M} = \varepsilon.$$

This holds for all $\varepsilon > 0$, so $\lim_{N\to\infty} A_d^N x_0 = 0$.
To prove the "$\Leftarrow$" implication, we choose an $\varepsilon > 0$. Since $\lim_{k\to\infty} A_d^k x_0 = 0$, there exists a n_ε such that for all $k > n_\varepsilon$, $\|A_d^k x_0\| < \varepsilon$.

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{N}\|A_d^k x_0\|^2 ds &= \frac{1}{N}\sum_{k=1}^{n_\varepsilon}\|A_d^k x_0\|^2 ds + \frac{1}{N}\sum_{k=n_\varepsilon+1}^{N}\|A_d^k x_0\|^2 ds\\ &\leq \frac{M^2\|x_0\|^2 n_\varepsilon}{N} + \varepsilon\frac{N-n_\varepsilon}{N}\end{aligned}$$

For N large enough, the left-hand side is smaller than 2ε. This holds for all $\varepsilon > 0$, so the left-hand side goes to 0, as N goes to ∞. □

We end this section with a characterization of the boundedness of an operator sequence.

Lemma 2.23 *Let A_d be a bounded operator on the Hilbert space X. Then the operator sequence $(A_d^n)_{n\geq 0}$ is bounded if and only if the following estimates hold for all $x \in X$:*

$$\sup_{r\in[0,1)}(1-r)\sum_{n=0}^{\infty}\|A_d^n x_0\| r^{2n} \leq M_1\|x_0\|^2, \tag{2.36}$$

$$\sup_{r\in[0,1)}(1-r)\sum_{n=0}^{\infty}\|A_d^{*n} x_0\| r^{2n} \leq M_2\|x_0\|^2. \tag{2.37}$$

Furthermore, if equations (2.36) and (2.37) hold, then the bound of $(A_d^n)_{n\geq 0}$ is given by

$$\|A_d^n\| \leq e\sqrt{M_1 M_2}.$$

For the proof we refer to Guo and Zwart [21, Theorem 3.2 and Remark 3.3].

2.3 From continuous to discrete time

In the previous sections we stated stability results for either continuous or discrete time systems. In this section we focus on Lyapunov equations which which show the interaction between these systems.
From Theorem 2.5 and Remark 2.7 we know that for every exponentially stable semigroup $(e^{At})_{t\geq 0}$, one can find a unique solution Q of the continuous Lyapunov equation,

$$A^*Q + QA = -I. \tag{2.38}$$

With this equation, we show that Q is also the solution to the discrete Lyapunov equation for A_d,

$$A_d^*QA_d - Q = -C^*C, \qquad \text{with } C = \sqrt{2}(A-I)^{-1}. \tag{2.39}$$

This discrete Lyapunov equation leads to the following estimate:

Lemma 2.24 (Lyapunov estimate) *Let A generate an exponentially stable C_0-semigroup, such that, $\|e^{At}\| \leq Me^{-\omega t}$ with $\omega > 0$, then for the Cayley transform A_d of A and the operator $C = \sqrt{2}(A-I)^{-1}$, the following estimate holds:*

$$\sum_{n=0}^{\infty} \|CA_d^n x\|^2 \leq \frac{M^2}{2\omega}\|x\|^2. \tag{2.40}$$

Proof: The semigroup $(e^{At})_{t\geq 0}$ is exponentially stable, so there exists a Lyapunov function Q such that:

$$A^*Q + QA = -I. \tag{2.41}$$

From equation (2.17) we know that $\|Q\| \leq \frac{M^2}{2\omega}$.
We show that for the Cayley transform A_d a discrete Lyapunov equation exists with solution Q. For this we use equation (2.41):

$$\begin{aligned} A_d^*QA_d - Q &= (A-I)^{-*}\Big[(A+I)^*Q(A+I) \\ &\qquad - (A-I)^*Q(A-I)\Big](A-I)^{-1} \\ &= C^*\Big[A^*Q + QA\Big]C \\ &= -C^*C. \end{aligned} \tag{2.42}$$

From this and equation (2.32), the following estimate for A_d follows:

$$\sum_{n=0}^{\infty} \|CA_d^n x\|^2 \leq \|Q\|\|x\|^2 \leq \frac{M^2}{2\omega}\|x\|^2.$$

This proves the lemma. □

For further details on Lyapunov equations we refer to [10, Chapter 4].

Remark 2.25 *For A^*, a similar estimate holds:*

$$\sum_{n=0}^{\infty} \|C^* A_d^{*n} x\|^2 \leq \frac{M^2}{2\omega} \|x\|^2. \tag{2.43}$$

We would like to point at the importance of the operator C. If $C = I$, Lemma 2.24 would prove stability of A_d.

Remark 2.26 *The operator Q is a solution of continuous Lyapunov equation (2.38) and discrete Lyapunov equation (2.42). By Remark 2.8 Q satisfies equation (2.16) By Lemma 2.19 Q satisfies inequality (2.32) as well. Combining these relations, gives the following result:*

$$\sum_{n=0}^{\infty} 2\|A_d^n (A - I)^{-1} x_0\|^2 \leq \int_0^{\infty} \|e^{At} x_0\|^2 dt. \tag{2.44}$$

Equality holds as well, this is a special case of Theorem 6.11.

We introduce the following notation $A^{-*} = \left(A^{-1}\right)^*$.
Now we focus on three specific Lyapunov equations and the relation between them. These equations are important in Chapter 7, where we look at growth of the semigroup generated by the inverse of A as well.
Under the assumption that λ is real and λ, λ^{-1} are in the resolvent set of A, we consider the following two Lyapunov equations

$$(\lambda^2 - 1)(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1} - P_1 = -I. \tag{2.45}$$

and

$$(\lambda^2 - 1)(\lambda I - A^{-1})^{-*} P_2 (\lambda I - A^{-1})^{-1} - P_2 = -I. \tag{2.46}$$

For the Cayley transform we consider the following Lyapunov equation

$$\left(\frac{\lambda - 1}{\lambda + 1}\right)(A - I)^{-*}(A + I)^* P_V (A + I)(A - I)^{-1} - P_V = -I. \tag{2.47}$$

Lemma 2.27 *Let $\lambda \in \mathbb{R}$ be such that λ and λ^{-1} are in $\rho(A)$, the resolvent set of A. Furthermore, assume that $1 \in \rho(A)$.*

1. *The Lyapunov equation (2.45) has a solution if and only if (2.46) has a solution. Furthermore, the solutions are related via*

$$P_2 = (I - \lambda A)^* (\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1} (I - \lambda A). \tag{2.48}$$

2. If (2.45) has a solution, then a solution of (2.47) is given by

$$\frac{1}{2\lambda}(P_1 + P_2 + \lambda I - I) = P_V. \tag{2.49}$$

Proof:
Part 1. Equation (2.45) can be equivalently written as

$$\begin{aligned}
(\lambda^2 - 1)(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1} - P_1 &= -I \\
(\lambda^2 - 1) P_1 - (\lambda I - A)^* P_1 (\lambda I - A) &= -(\lambda I - A)^*(\lambda I - A) \\
-A^* P_1 A + \lambda A^* P_1 + \lambda P_1 A - P_1 &= -(\lambda I - A)^*(\lambda I - A)
\end{aligned} \tag{2.50}$$

Equation (2.46) can be equivalently written as

$$\begin{aligned}
(\lambda^2 - 1)(\lambda I - A^{-1})^{-*} P_2 (\lambda I - A^{-1})^{-1} - P_2 &= -I \\
(\lambda^2 - 1) P_2 - (\lambda I - A^{-1})^* P_2 (\lambda I - A^{-1}) &= -(\lambda I - A^{-1})^*(\lambda I - A^{-1}) \\
-A^{-*} P_2 A^{-1} + \lambda A^{-*} P_2 + \lambda P_2 A^{-1} - P_2 &= -(\lambda I - A^{-1})^*(\lambda I - A^{-1}) \\
-P_2 + \lambda P_2 A + \lambda A^* P_2 - A^* P_2 A &= -(\lambda A - I)^*(\lambda A - I).
\end{aligned} \tag{2.51}$$

Assuming that P_1 satisfies (2.50), we substitute $(I - \lambda A)^*(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1}(I - \lambda A)$ in the left hand-side of (2.51),

$$\begin{aligned}
&- (I - \lambda A)^*(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1}(I - \lambda A) + \\
&\quad \lambda (I - \lambda A)^*(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1}(I - \lambda A) A + \\
&\quad\quad \lambda A^* (I - \lambda A)^*(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1}(I - \lambda A) - \\
&\quad\quad\quad A^* (I - \lambda A)^*(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1}(I - \lambda A) A \\
&= (I - \lambda A)^*(\lambda I - A)^{-*} \left[-P_1 + \lambda A P_1 + \lambda P_1 A^* - A^* P_1 A \right] \\
&\quad (\lambda I - A)^{-1}(I - \lambda A) \\
&= (I - \lambda A)^*(\lambda I - A)^{-*} \left[-(\lambda I - A)^*(\lambda I - A) \right] (\lambda I - A)^{-1}(I - \lambda A) \\
&= -(I - \lambda A)^*(I - \lambda A).
\end{aligned}$$

Thus we see that $(I - \lambda A)^*(\lambda I - A)^{-*} P_1 (\lambda I - A)^{-1}(I - \lambda A)$ satisfies equation (2.51) and thus it is a solution of (2.46). The other implication is proved similarly.

Part 2. Since (2.45) has a solution, we have by part 1. that (2.46) also has a solution.

The left hand-side of the Lyapunov equation (2.47) can written as

$$
\begin{aligned}
&\left(\frac{\lambda-1}{\lambda+1}\right)(A-I)^{-*}(A+I)^* P_V (A+I)(A-I)^{-1} - P_V \\
&\quad= \frac{1}{\lambda+1}(A-I)^{-*}((\lambda-1)(A+I)^* P_V (A+I) \\
&\qquad\qquad\qquad\qquad - (\lambda+1)(A-I)^* P_V (A-I))(A-I)^{-1} \\
&\quad= \frac{2}{\lambda+1}(A-I)^{-*}(-A^* P_V A + \lambda A^* P_V + \lambda P_V A - P_V)(A-I)^{-1}.
\end{aligned}
\tag{2.52}
$$

Thus the Lyapunov equation (2.47) can be equivalently written as

$$
-A^* P_V A + \lambda A^* P_V + \lambda P_V A - P_V = -\frac{\lambda+1}{2}(A-I)^*(A-I). \tag{2.53}
$$

Next we replace P_V in the left hand-side of this equation by $P_1+P_2+\lambda I - I$, and we obtain

$$
\begin{aligned}
-A^*(P_1&+P_2+\lambda I - I)A + \lambda A^*(P_1+P_2+\lambda I - I) + \\
&\lambda(P_1+P_2+\lambda I - I)A - (P_1+P_2+\lambda - I) \\
&= -(\lambda I - A)^*(\lambda I - A) - (\lambda A - I)^*(\lambda A - I) \\
&\quad + (\lambda-1)(-A^*A + \lambda A^* + \lambda A - I) \\
&= -\lambda^2 I + \lambda A^* + \lambda A - A^*A - \lambda^2 A^*A + \lambda A^* + \lambda A - I + \\
&\quad \lambda(-A^*A + \lambda A^* + \lambda A - I) + A^*A - \lambda A^* - \lambda A + I \\
&= (-\lambda^2 - \lambda)(A^*A - A^* - A + I),
\end{aligned}
$$

where we have used (2.50) and (2.51). Thus if we choose $P_V = \frac{1}{2\lambda}(P_1 + P_2 + \lambda I - I)$, then the left hand-side of (2.53) becomes

$$
\frac{1}{2\lambda}(-\lambda^2 - \lambda)(A^*A - A^* - A + I) = -\frac{\lambda+1}{2}(A-I)^*(A-I). \tag{2.54}
$$

Since this equals the right hand-side of (2.53) we see that $\frac{1}{2\lambda}(P_1+P_2+\lambda I - I)$ is a solution of the Lyapunov equation (2.47). □

This ends Chapter 2. In this chapter we summarized stability results for continuous and discrete time systems.

Chapter 3

Log estimate using Lyapunov equations

3.1 Overview

In Chapter 1 we have seen that for a bounded semigroup on a Banach space the powers of the corresponding cogenerator do not grow faster that $\sqrt{n}$. In [20], Gomilko proved that for bounded semigroups on a Hilbert space this estimate can be improved to $\ln(n+1)$. In this chapter, we obtain a similar result. However, only for exponentially stable semigroups. Our proof is very different than that of Gomilko and it is based on Lyapunov equations. The result is:

Theorem 3.1 *Let operator A generate the exponentially stable C_0-semigroup $\left(e^{At}\right)_{t\geq 0}$ on the Hilbert space X. Furthermore, let M and $\omega > 0$ be such that $\|e^{At}\| \leq Me^{-\omega t}$. Then for the n-th power of its cogenerator A_d, the following estimate holds:*

$$\|A_d^n\| \leq 1 + 2M + \left(\frac{M}{\sqrt{2}} + 1\right)\frac{M}{\sqrt{\omega}} + ({}_2\log n - 1)\left(M^2 + \sqrt{2}M\right). \quad (3.1)$$

The most important term on the right-hand side is the ${}_2\log n$-term. This is the part that depends on n and indicates the growth of $\|A_d^n\|$ as $n \to \infty$. In the ${}_2\log n$-term depends quadratically on M. This is the same as in the proof of Gomilko, where the $\ln(n+1)$-term depends quadratically on M, the bound of the semigroup.

As mentioned above our proof uses Lyapunov equations. This technique is also used by Zwart for estimating the growth of $\left(e^{A^{-1}t}\right)_{t\geq 0}$, see [39]. Although the proof is rather technical, the idea can easily be explained. Let $k \in \mathbb{N}$ and $n = 2^k$. For A_d we introduce a sequence of operators $A_0, A_1, ..., A_k$ such that $A_0 = A_d$ and the norm

$$\|A_j^{2m} - A_{j+1}^m\| \leq M^2, \qquad m \in \mathbb{N}.$$

Using this equation, we find

$$\begin{aligned}
\|A_0^n - A_k\| &= \|\sum_{j=0}^{k-1} A_j^{n_j} - A_{j+1}^{n_{j+1}}\|, \qquad n_j = \frac{n}{2^j} \\
&\leq \sum_{j=0}^{k-1} \|A_j^{n_j} - A_{j+1}^{n_{j+1}}\|, \\
&\leq \sum_{j=0}^{k-1} M^2 = kM^2 = {}_2\log n \; M^2.
\end{aligned}$$

For A_k we derive an estimate which is independent of k. Thus equation (3.1) is derived for n a power of 2. Obtaining equation (3.1) for all n, requires an extra inequality, see Lemma 3.5.
The $\sqrt{n}$-bound in Banach spaces is a sharp bound. Example 1.2 shows a contraction semigroup of which the powers of the cogenerator grow like $\sqrt{n}$. For the $\ln(n+1)$-bound in Hilbert spaces it is not known whether the bound is sharp. Moreover, under the conditions of Theorem 3.1 there is no example known for which $\|A_d^n\|$ is unbounded.
In Section 3.2 we define the sequence $\{A_k\}_{k\in\mathbb{N}}$ and we prove the above mentioned estimates. Thus this becomes a quite technical section.
In Section 3.3 the proof of Theorem 3.1 is given.

3.2 Estimates on operators

In this section we apply the Lyapunov estimates of Section 2.3, to show the technical steps towards the proof of Theorem 3.1.
Firstly, we define a sequence of operators which starts with A_d^n. Secondly, we estimate the norm of the last operator in this sequence. Finally, we estimate the difference between two adjacent operators in the sequence. By repeatedly applying this last estimate, we can prove the main result as indicated in the previous section. This will be done in Section 3.3.
We begin by defining a sequence of cogenerators and output operators.

Definition 3.2 *For A given in Theorem 3.1 we define the operators A_j and C_j by:*

$$A_j := (\gamma_j A - \varepsilon_j I + I)\,(\gamma_j A - \varepsilon_j I - I)^{-1}\,, \tag{3.2}$$

$$C_j = \sqrt{2}\,(\gamma_j A - \varepsilon_j I - I)^{-1}\,, \qquad j = 0, 1, ... \tag{3.3}$$

with:

$$\gamma_{j+1} = \frac{1}{2}\gamma_j, \qquad \gamma_0 = 1 \tag{3.4}$$

$$\varepsilon_{j+1} = \frac{1}{2} + \frac{1}{2}\varepsilon_j, \qquad \varepsilon_0 = 0 \tag{3.5}$$

We make several remarks concerning this definition.

Remark 3.3

- *$A_0 = A_d$.*
- *A_j is the Cayley transform of $\gamma_j A - \varepsilon_j I$. Since $\gamma_j, \varepsilon_j \geq 0$, we have that $\gamma_j A - \varepsilon_j I$ generates an exponentially stable semigroup on X. Hence, we can apply Lemma 2.24 to C_j, and $\gamma_j A - \varepsilon_j I$.*

Now, we consider the operator sequence:

$$A_j^{n_j}, \qquad \text{with } j \in \{0, \ldots, N\}, \quad N = \lfloor {}_2\log n \rfloor, \tag{3.6}$$

$$n_{j+1} = \lfloor \frac{n_j}{2} \rfloor, \qquad \text{with } n_0 = n, \quad n_N = 1. \tag{3.7}$$

Here $\lfloor \ell \rfloor$ is the largest integer not greater than ℓ, which is called the floor function. We start with an estimate on the norm of the last operator, $A_N^{n_N} = A_N$:

Lemma 3.4 *Let A_N be the operator as defined by Definition 3.2, then the following estimate holds:*

$$\|A_N\| \leq 1 + 2M. \tag{3.8}$$

Proof: For the proof, we use the Hille-Yosida Theorem [10, Theorem 2.1.12] and the fact that γ_N, ω, and ε_N are positive:

$$\begin{aligned}
\|A_N\| &= \|\,(\gamma_N A - \varepsilon_N I + I)\,(\gamma_N A - \varepsilon_N I - I)^{-1}\,\| \\
&= \|\,(\gamma_N A - \varepsilon_N I - I + 2I)\,(\gamma_N A - \varepsilon_N I - I)^{-1}\,\| \\
&= \|I + 2\,(\gamma_N A - (\varepsilon_N + 1)I)^{-1}\,\| \\
&\leq 1 + 2\frac{M}{\gamma_N \omega + \varepsilon_N + 1} \leq 1 + 2M.
\end{aligned}$$

This proves the lemma. □

Next, we estimate the difference between two successive operators, $A_j^{n_j}$ and $A_{j+1}^{n_{j+1}}$. For this, we need the following lemma, which focusses on even n_j.

Lemma 3.5 *Let A_k and A_{k+1} be defined by Definition 3.2, and let $m \in \mathbb{N}$. Then the following estimate holds:*

$$\|A_k^{2m} - A_{k+1}^m\| \leq \frac{M^2}{2\sqrt{\gamma_k\omega + \varepsilon_k}\sqrt{\gamma_{k+1}\omega + \varepsilon_{k+1}}}. \tag{3.9}$$

Proof: Firstly, we prove the result for $m = 1$.

$$\begin{aligned}
A_k^2 &- A_{k+1} \\
&= (\gamma_k A - \varepsilon_k I + I)^2 (\gamma_k A - \varepsilon_k I - I)^{-2} \\
&\quad - (\gamma_{k+1} A - \varepsilon_{k+1} I + I)(\gamma_{k+1} A - \varepsilon_{k+1} I - I)^{-1} \\
&= (\gamma_k A - \varepsilon_k I - I)^{-2} \cdot \\
&\quad \Big[(\gamma_k A - \varepsilon_k I + I)^2 (\gamma_{k+1} A - \varepsilon_{k+1} I - I) \\
&\quad - (\gamma_k A - \varepsilon_k I - I)^2 (\gamma_{k+1} A - \varepsilon_{k+1} I + I) \Big] \cdot \\
&\quad (\gamma_{k+1} A - \varepsilon_{k+1} I - I)^{-1}
\end{aligned} \tag{3.10}$$

We simplify the middle part:

$$\begin{aligned}
(\gamma_k A - &\varepsilon_k I + I)^2 (\gamma_{k+1} A - \varepsilon_{k+1} I - I) \\
&- (\gamma_k A - \varepsilon_k I - I)^2 (\gamma_{k+1} A - \varepsilon_{k+1} I + I) \\
&= (\gamma_k A - \varepsilon_k I)^2 (\gamma_{k+1} A - \varepsilon_{k+1} I) + 2(\gamma_k A - \varepsilon_k I)(\gamma_{k+1} A - \varepsilon_{k+1} I) \\
&\quad + (\gamma_{k+1} A - \varepsilon_{k+1} I) - (\gamma_k A - \varepsilon_k I)^2 - 2(\gamma_k A - \varepsilon_k I) - I \\
&\quad - (\gamma_k A - \varepsilon_k I)^2 (\gamma_{k+1} A - \varepsilon_{k+1} I) + 2(\gamma_k A - \varepsilon_k I)(\gamma_{k+1} A - \varepsilon_{k+1} I) \\
&\quad - (\gamma_{k+1} A - \varepsilon_{k+1} I) - (\gamma_k A - \varepsilon_k I)^2 + 2(\gamma_k A - \varepsilon_k I) - I \\
&= 4(\gamma_k A - \varepsilon_k I)(\gamma_{k+1} A - \varepsilon_{k+1} I) - 2(\gamma_k A - \varepsilon_k I)^2 - 2I \\
&= \left(4\gamma_k\gamma_{k+1} - 2\gamma_k^2\right) A^2 + \left(4\varepsilon_k\gamma_k - 4\varepsilon_k\gamma_{k+1} - 4\varepsilon_{k+1}\gamma_k\right) A \\
&\quad + \left(4\varepsilon_k\varepsilon_{k+1} - 2\varepsilon_k^2 - 2\right) I \\
&= \left(2\varepsilon_k - 4\varepsilon_{k+1}\right)\gamma_k A + \left(4\varepsilon_k\varepsilon_{k+1} - 2\varepsilon_k^2 - 2\right) I \\
&= -2\gamma_k A + (2\varepsilon_k - 2) I \\
&= -2(\gamma_k A - \varepsilon_k I + I),
\end{aligned}$$

where we used equation (3.4) and (3.5). Substituting this in equation (3.10), and using equation (3.3), gives:

$$\begin{aligned} A_k^2 - A_{k+1} & \\ &= -2\left(\gamma_k A - \varepsilon_k I - I\right)^{-2}\left(\gamma_k A - \varepsilon_k I + I\right)\left(\gamma_{k+1} A - \varepsilon_{k+1} I - I\right)^{-1} \\ &= -A_k C_k C_{k+1}. \end{aligned} \tag{3.11}$$

Secondly, we remark that we can write $A_k^{2m} - A_{k+1}^m$ as the following finite sum,

$$\begin{aligned} A_k^{2m} - A_{k+1}^m &= \sum_{j=0}^{m-1} A_k^{2(m-j)} A_{k+1}^j - A_k^{2(m-1-j)} A_{k+1}^{j+1} \\ &= \sum_{j=0}^{m-1} A_k^{2(m-1-j)} \Big[A_k^2 - A_{k+1}\Big] A_{k+1}^j. \end{aligned} \tag{3.12}$$

Now using equation (3.11) and equation (3.12), we get the following expression:

$$\begin{aligned} \|A_k^{2m} - A_{k+1}^m\| &= \|\sum_{j=0}^{m-1} A_k^{2(m-1-j)} \Big[A_k^2 - A_{k+1}\Big] A_{k+1}^j\| \\ &= \|\sum_{j=0}^{m-1} A_k^{2(m-j)-1} C_k\; C_{k+1} A_{k+1}^j\| \\ &= \sup_{\|x\|=1} \|\sum_{j=0}^{m-1} A_k^{2(m-j)-1} C_k\; C_{k+1} A_{k+1}^j x\| \\ &= \sup_{\|x\|,\|y\|=1} |\langle y, \sum_{j=0}^{m-1} A_k^{2(m-j)-1} C_k\; C_{k+1} A_{k+1}^j x\rangle| \\ &\leq \sup_{\|x\|,\|y\|=1} \sum_{j=0}^{m-1} |\langle y, A_k^{2(m-j)-1} C_k\; C_{k+1} A_{k+1}^j x\rangle| \\ &= \sup_{\|x\|,\|y\|=1} \sum_{j=0}^{m-1} |\langle C_k^* A_k^{*2(m-j)-1} y, C_{k+1} A_{k+1}^j x\rangle| \end{aligned}$$

$$\begin{aligned}
\|A_k^{2m} - A_{k+1}^m\| & \\
&\leq \sup_{\|x\|,\|y\|=1} \left(\sum_{j=0}^{m-1} \|C_k^* A_k^{*2(m-j)-1} y\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{m-1} \|C_{k+1} A_{k+1}^j x\|^2 \right)^{\frac{1}{2}} \\
&\leq \sup_{\|x\|,\|y\|=1} \left(\sum_{j=0}^{\infty} \|C_k^* A_k^{*j} y\|^2 \right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} \|C_{k+1} A_{k+1}^j x\|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{M^2}{2\sqrt{\gamma_k \omega + \varepsilon_k}\sqrt{\gamma_{k+1}\omega + \varepsilon_{k+1}}}.
\end{aligned}$$

In the second last step we used the Cauchy-Schwarz inequality on $\ell_2(X)$. In the last step we used Lemma 2.24 and Remark 2.25 and 3.3. □

If n_k, see equation (3.7), is even, then Lemma 3.5 gives an estimate for two successive operators. For an odd power n_k, we need an extra step.

Lemma 3.6 *For A_k defined by Definition 3.2 the following estimate holds:*

$$\|A_k^{n+1} - A_k^n\| \leq \frac{M}{\sqrt{\gamma_k \omega + \varepsilon_k}}. \tag{3.13}$$

Proof: Using the definition of C_k and A_k, we find that:

$$\begin{aligned}
\|A_k^{n+1} - A_k^n\| &= \| (A_k - I) A_k^n\| = \| \left(I + \sqrt{2} C_k - I \right) A_k^n \| \\
&= \sqrt{2} \|C_k A_k^n\| = \sup_{x \neq 0} \sqrt{2} \frac{\|C_k A_k^n x\|}{\|x\|} \\
&\leq \sup_{x \neq 0} \left(2 \sum_{n=0}^{\infty} \frac{\|C_k A_k^n x\|^2}{\|x\|^2} \right)^{\frac{1}{2}} \leq \frac{M}{\sqrt{\gamma_k \omega + \varepsilon_k}},
\end{aligned}$$

where we have used Lemma 2.24, see also Remark 3.3. □

The previous three lemma's enable us to estimate the difference between $A_k^{n_k}$ and $A_{k+1}^{n_{k+1}}$. The difference between two successive operators can be estimated as follows:

Lemma 3.7 *Let A_k and A_{k+1} be defined by Definition 3.2 and let n_k, n_{k+1} be defined in (3.7). The following estimate holds:*

$$\|A_k^{n_k} - A_{k+1}^{n_{k+1}}\| \leq \frac{M^2}{2\sqrt{\gamma_k \omega + \varepsilon_k}\sqrt{\gamma_{k+1}\omega + \varepsilon_{k+1}}} + \frac{M}{\sqrt{\gamma_k \omega + \varepsilon_k}}. \tag{3.14}$$

Proof: If n_k is even, then estimate (3.9) of Lemma 3.5 implies equation (3.14). In case n_k is odd, we combine Lemma 3.5 and Lemma 3.6 to obtain

$$\begin{aligned}
\|A_k^{n_k} - A_{k+1}^{n_{k+1}}\| &= \|A_k^{n_k} - A_{k+1}^{\frac{n_k-1}{2}}\| \\
&= \|A_k^{n_k} - A_k^{n_k-1} + A_k^{n_k-1} - A_{k+1}^{\frac{n_k-1}{2}}\| \\
&\leq \|A_k^{n_k} - A_k^{n_k-1}\| + \|A_k^{n_k-1} - A_{k+1}^{\frac{n_k-1}{2}}\| \\
&\leq \frac{M}{\sqrt{\gamma_k\omega+\varepsilon_k}} + \frac{M^2}{2\sqrt{\gamma_k\omega+\varepsilon_k}\sqrt{\gamma_{k+1}\omega+\varepsilon_{k+1}}}.
\end{aligned}$$

Thus we have proved equation (3.14). □

As explained in Section 3.1, these estimates will give us the inequality (3.1).

3.3 Proof of Theorem 3.1

In this section we prove Theorem 3.1. For this we use the notation of Definition 3.2 and the estimates of Lemma 3.4 and Lemma 3.7.

Proof of Theorem 3.1: We write A_d^n as a sum of operators of equation (3.6):

$$\begin{aligned}
\|A_d^n\| = \|A_0^{n_0}\| &= \|A_N + (A_0^{n_0} - A_1^{n_1}) + \sum_{j=1}^{N-1}(A_j^{n_j} - A_{j+1}^{n_{j+1}})\| \\
&\leq \|A_N\| + \|A_0^{n_0} - A_1^{n_1}\| + \sum_{j=1}^{N-1}\|A_j^{n_j} - A_{j+1}^{n_{j+1}}\|. \qquad (3.15)
\end{aligned}$$

By Lemma 3.4, we have that.

$$\|A_N\| \leq 1 + 2M. \qquad (3.16)$$

Furthermore, by Lemma 3.7 we have that for $k = 0, 1, ...$

$$\|A_k^{n_k} - A_{k+1}^{n_{k+1}}\| \leq \frac{M^2}{2\sqrt{\gamma_k\omega+\varepsilon_k}\sqrt{\gamma_{k+1}\omega+\varepsilon_{k+1}}} + \frac{M}{\sqrt{\gamma_k\omega+\varepsilon_k}}. \qquad (3.17)$$

Combining equation (3.15), (3.16), and (3.17), we find that

$$\begin{aligned}
\|A_d^n\| \leq 1 + 2M &+ \frac{M^2}{2\sqrt{\gamma_0\omega+\varepsilon_0}\sqrt{\gamma_1\omega+\varepsilon_1}} + \frac{M}{\sqrt{\gamma_0\omega+\varepsilon_0}} \\
&+ \sum_{j=1}^{N-1}\left(\frac{M^2}{2\sqrt{\gamma_k\omega+\varepsilon_k}\sqrt{\gamma_{k+1}\omega+\varepsilon_{k+1}}} + \frac{M}{\sqrt{\gamma_k\omega+\varepsilon_k}}\right).
\end{aligned}$$

Since $\gamma_0\omega + \varepsilon_0 = \omega$, $(\gamma_k\omega + \varepsilon_k)^{-\frac{1}{2}} \leq \sqrt{2}$ for $k \geq 1$, and $N = \lfloor {}_2\log n \rfloor$, we can majorize the above by

$$\begin{aligned}
\|A_d^n\| &\leq 1 + 2M + \frac{M^2}{\sqrt{2\omega}} + \frac{M}{\sqrt{\omega}} + \sum_{j=1}^{N-1} \left(M^2 + \frac{M}{\sqrt{2}} \right) \\
&= 1 + 2M + \left(\frac{M}{\sqrt{2}} + 1 \right) \frac{M}{\sqrt{\omega}} + ({}_2\log n - 1) \left(M^2 + \sqrt{2}M \right).
\end{aligned}$$

Thus we have proved inequality (3.1) and thereby Theorem 3.1. □

We remark that for this estimate ω has to be positive. As ω approaches zero, we can see from the $\sqrt{\omega}^{-1}$ term on the right-hand side of inequality (3.1), the estimate is getting worse.
This means that the semigroup has a negative growth bound, and the estimate does not hold for bounded semigroups.

3.4 Conclusions

In this chapter we provided a logarithmic estimate for the powers of the cogenerator of an exponentially stable semigroup on a Hilbert space. For the proof we used an estimate with Lyapunov functions and a technical method which divided the estimate in $\ln(n+1)$ sub-estimates. A similar estimate for bounded semigroups was proved by Gomilko, [20].

Chapter 4

Bergman distance

4.1 Introduction

In Chapter 1 we have seen that for contraction semigroups and bounded analytic semigroups the power sequence of the corresponding cogenerator is bounded. In Chapter 3 we showed that for a general exponentially stable semigroup, the growth of its cogenerator is at most logarithmic. In this chapter we show that if two semigroups are close to each other, then the corresponding cogenerators share the same growth properties.
We measure the distance between two semigroups in the Bergman distance.

Definition 4.1 *The semigroups, $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ on the Hilbert space X, have a* finite Bergman distance *if the following two inequalities are satisfied for all $x_0 \in X$:*

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt < \infty, \tag{4.1}$$

$$\int_0^\infty \|(e^{A^*t} - e^{\tilde{A}^*t})x_0\|^2 \frac{1}{t} dt < \infty. \tag{4.2}$$

Note that the measure $t^{-1}dt$ is the invariant measure for the multiplication group $\mathbb{R}^+$. The space $L^2_1(\mathbb{R}^+)$ with this measure is isometrically isomorphic to the unweighted Bergman space $\mathcal{A}^2(\Pi^+)$, see [14, Theorem 1]. Thus two semigroups have finite Bergman distance, if $(e^{At} - e^{\tilde{A}t})x_0$ and $(e^{A^*t} - e^{\tilde{A}^*t})x_0$ are in the Bergman space for all $x_0 \in X$. Hence the name Bergman distance.

Remark 4.2 *Assume that the integral term in equation (4.1) is finite for all $x_0 \in X$. From the Uniform Boundedness Theorem, we know this is equivalent to the existence of a $b \in [0,\infty)$ such that*

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt \le b^2 \|x_0\|^2, \qquad \textit{for all } x_0 \in X. \tag{4.3}$$

Proof: Let (Q_n) be a sequence of linear operator $Q_n : X \to L^2(0,\infty)$ defined by

$$Q_n x_0 = (e^{At} - e^{\tilde{A}t})x_0 \frac{\mathbf{1}_{[\frac{1}{n},n]}(t)}{\sqrt{t}},$$

where $\mathbf{1}_{[\frac{1}{n},n]}$ is the indication function of $[\frac{1}{n}, n]$.
Using inequality (4.1), we know that for every $x_0 \in X$ there exists a constant M_{x_0} such that

$$\begin{aligned} \|Q_n x_0\|^2 &= \int_{\frac{1}{n}}^n \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt \\ &\le \int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt = M_{x_0}, \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

Using the Uniform Boundedness Theorem, we find that there exists a constant b such that

$$\|Q_n\| \le b, \qquad \text{for all } n \in \mathbb{N}.$$

Hence,

$$\lim_{n\to\infty} \|Q_n x_0\|^2 = \int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt \le b^2 \|x_0\|^2.$$

Thus inequality (4.3) holds. □

A similar remark holds for inequality (4.2).
Let $\mathbb{B}$ be the set of all $b \in [0,\infty)$ for which equation (4.3) holds. It is clear that $\mathbb{B}$ is a subinterval of $[0,\infty)$.

Definition 4.3 *Let the semigroups $(e^{At})_{t\ge0}$ and $(e^{\tilde{A}t})_{t\ge0}$ satisfy equation (4.1) and (4.2) and let $\mathbb{B}$ be the set of all $b \in [0,\infty)$ for which equation (4.3) holds. The* Bergman distance *between the semigroups, $(e^{At})_{t\ge0}$ and $(e^{\tilde{A}t})_{t\ge0}$, is defined as*

$$d\left(e^{At}, e^{\tilde{A}t}\right) = \inf_{b\in\mathbb{B}} b. \tag{4.4}$$

So two semigroups have a finite Bergman distance if and only if $d\left(e^{At}, e^{\tilde{A}t}\right) < \infty$.
In Chapter 5, we show that $d\left(e^{At}, e^{\tilde{A}t}\right)$ defines a metric on the space of C_0-semigroups and that it divides the space of C_0-semigroups on X into equivalence classes. In this chapter we study properties of C_0-semigroups with a finite Bergman distance. In Section 4.2 we show that the semigroups with finite Bergman distance share the same stability properties. In Section 4.6, we investigate which pair of generators have a finite Bergman distance. Among others, we show that if A and $\tilde{A}$ generate exponentially stable semigroups, and if $A-\tilde{A}$ is a bounded operator, then they have a finite Bergman distance.
As to infinitesimal generators, we define a *discrete-time Bergman distance.*

Definition 4.4 *The cogenerators, $(A_d^n)_{n\in\mathbb{N}}$ and $(\tilde{A}_d^n)_{n\in\mathbb{N}}$, have a* finite Bergman distance *if the following two inequalities are satisfied for all $x_0 \in X$:*

$$\sum_{k=1}^{\infty} \frac{1}{k}\|(A_d^k - \tilde{A}_d^k)x_0\|^2 < \infty, \tag{4.5}$$

$$\sum_{k=1}^{\infty} \frac{1}{k}\|(A_d^{*k} - \tilde{A}_d^{*k})x_0\|^2 < \infty. \tag{4.6}$$

Remark 4.5 *Assume that the sum in equation (4.5) is finite for all $x_0 \in X$. From the Uniform Boundedness Theorem, this is equivalent to the existence of a $b \in [0,\infty)$ such that*

$$\sum_{k=1}^{\infty} \frac{1}{k}\|(A_d^k - \tilde{A}_d^k)x_0\|^2 \leq b\|x_0\|^2, \qquad \text{for all } x_0 \in X. \tag{4.7}$$

The proof of this result is similar to the proof of Remark 4.2.
A similar remark holds for inequality (4.6).
Let $\mathbb{B}$ be the set of all $b \in [0,\infty)$ for which equation (4.7) holds. It is clear that $\mathbb{B}$ is a subinterval of $[0,\infty)$.

Definition 4.6 *Let the cogenerators $(A_d^n)_{n\in\mathbb{N}}$ and $(\tilde{A}_d^n)_{n\in\mathbb{N}}$ satisfy equation (4.5) and (4.6) and let $\mathbb{B}$ be the set of all $b \in [0,\infty)$ for which equation (4.7) holds. The* Bergman distance *between the cogenerators, $(A_d^n)_{n\in\mathbb{N}}$ and $(\tilde{A}_d^n)_{n\in\mathbb{N}}$, is defined as*

$$d\left(A_d^n, \tilde{A}_d^n\right) = \inf_{b\in\mathbb{B}} b. \tag{4.8}$$

So two cogenerators have a finite Bergman distance if and only if $d\left(A_d^n, \tilde{A}_d^n\right) < \infty$.

As for semigroups, two cogenerators with finite Bergman distance share the same stability properties, see Section 4.3.
One of the main results of this chapter is, that the Cayley transform conserves the Bergman distance. That is, the following equality holds for all $x_0 \in X$:

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt = \sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - \tilde{A}_d^k)x_0\|^2. \tag{4.9}$$

This equality is proved in Section 4.4.
Combining the invariance of the stability properties with equation (4.9), leads to the following theorem:

Theorem 4.7 *Let $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance. Then $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ share the same stability properties, e.g. $(A_d^n)_{n\geq 0}$ is strongly stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable.*

Furthermore, the other implication also holds. Thus, if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have a finite Bergman distance, then $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have similar stability properties. We prove this in Section 4.5.

4.2 Properties of semigroups with finite Bergman distances

The finite Bergman distance groups semigroups into classes, see Section 4.1. In this section we show that within these classes of semigroups the stability properties are the same. For this we use the Van Casteren characterization of stability, see Lemma 2.9 and 2.10.

Theorem 4.8 *Let $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ be C_0-semigroups on the Hilbert space X having a finite Bergman distance. Then the following holds*

1. *$(e^{At})_{t\geq 0}$ is bounded if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is bounded,*
2. *$(e^{At})_{t\geq 0}$ is exponentially stable if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is exponentially stable,*
3. *$(e^{At})_{t\geq 0}$ is strongly stable if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is strongly stable.*

Proof: We prove the boundedness or stability of $(e^{At})_{t\geq 0}$, given the boundedness or stability of $(e^{\tilde{A}t})_{t\geq 0}$. By symmetry, the other implication then also holds. We begin with item 1.

1. For all $t > 0$ and $x_0 \in X$, the following inequalities hold:

$$\begin{aligned}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds &\leq \frac{1}{t}\int_0^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds + \frac{1}{t}\int_0^t 2\|e^{\tilde{A}s}x_0\|^2 ds \\ &\leq 2\int_0^t \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds + 2\sup_t \|e^{\tilde{A}t}\|^2\|x_0\|^2 \\ &\leq M_1\|x_0\|^2,\end{aligned}$$

   where we have used (4.1) and the boundedness of $(e^{\tilde{A}t})_{t\geq 0}$. Similarly, we obtain the dual result. Hence by Lemma 2.9, we conclude that $(e^{At})_{t\geq 0}$ is bounded.

2. For $t > 1$, we have for all $x_0 \in X$

$$\begin{aligned}\int_1^t \frac{1}{s}\|e^{As}x_0\|^2 ds &\leq 2\int_1^t \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds + 2\int_1^t \frac{1}{s}\|e^{\tilde{A}s}x_0\|^2 ds \\ &\leq M_2\|x_0\|^2, \end{aligned} \tag{4.10}$$

   where we have used the finite Bergman distance and the exponential stability of $(e^{\tilde{A}t})_{t\geq 0}$, see Remark 4.2.

   The exponential stability of $(e^{\tilde{A}t})_{t\geq 0}$ trivially implies that $(e^{\tilde{A}t})_{t\geq 0}$ is bounded. By item 1., we have that $(e^{At})_{t\geq 0}$ is bounded as well. Combining this with (4.10), we find for $t > 1$:

$$\begin{aligned}\ln(t)\|e^{At}x_0\|^2 &= \int_1^t \frac{1}{s}\|e^{At}x_0\|^2 ds \\ &\leq \int_1^t \frac{1}{s}\|e^{A(t-s)}\|^2\|e^{As}x_0\|^2 ds \\ &\leq M_1\int_1^t \frac{1}{s}\|e^{As}x_0\|^2 ds \leq M_1M_2\|x_0\|^2.\end{aligned}$$

   So for $t > 1$ we have that

$$\|e^{At}\|^2 \leq \frac{M_1M_2}{\ln(t)}.$$

   Since the right-hand side is less than one for sufficiently large t, we have by Lemma 2.3 that $(e^{At})_{t\geq 0}$ is exponentially stable.

3. Let x_0 be an element of X. Since $\int_0^\infty \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds < \infty$, for every $\varepsilon > 0$, there exists a t_ε such that

$$\int_{t_\varepsilon}^\infty \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds < \varepsilon. \tag{4.11}$$

Furthermore, the following inequality holds

$$\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds \leq \frac{1}{t}\int_0^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds + \frac{1}{t}\int_0^t 2\|e^{\tilde{A}s}x_0\|^2 ds.$$

Using this inequality and Lemma 2.10, we have that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds &\leq \lim_{t\to\infty}\frac{1}{t}\int_0^{t_\varepsilon} 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds \\ &\quad + \lim_{t\to\infty}\frac{1}{t}\int_{t_\varepsilon}^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds + 0 \\ &\leq \lim_{t\to\infty}\frac{t_\varepsilon}{t}\int_0^{t_\varepsilon} \frac{2}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds \\ &\quad + \lim_{t\to\infty}\int_{t_\varepsilon}^t \frac{2}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds \leq 0 + 2\varepsilon,\end{aligned}$$

where we have used inequality (4.11).

Since this holds for all $\varepsilon > 0$, we have shown that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds = 0,$$

and so by Lemma 2.10, $(e^{At})_{t\geq 0}$ is strongly stable.

This third item concludes the proof. □

Note that the proof of Theorem 4.8 is done element-wise. So if the semigroup $(e^{At})_{t\geq 0}$ provides a stable solution for the initial condition x_0, then semigroup $(e^{\tilde{A}t})_{t\geq 0}$ provides a stable solution for the same initial condition as well. This is independent of their behaviour on other elements in X. We examine this property of the Bergman distance in Lemma 5.7.

4.3 Properties of cogenerators with finite Bergman distances

The discrete-time case is similar to the continuous-time case. The finite Bergman distance also creates equivalence classes of sequences of bounded operators, see Section 5.1. Elements within a class share the same stability properties, as is shown next.

Theorem 4.9 *Let $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ be a power sequence of bounded operators on the Hilbert space X. If they have a finite Bergman distance, then the following assertions hold:*

1. *$(A_d^n)_{n\geq 0}$ is bounded if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is bounded,*
2. *$(A_d^n)_{n\geq 0}$ is power stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is power stable,*
3. *$(A_d^n)_{n\geq 0}$ is strongly stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable.*

Proof: We prove the boundedness or stability of $(A_d^n)_{n\geq 0}$, given the boundedness or stability of $(\tilde{A}_d^n)_{n\geq 0}$. By symmetry, the other implication then also holds. The proofs are similar to the ones in the continuous time.

1. For all $N \geq 1$ and $x_0 \in X$, the following hold,

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^{N} \|A_d^k x_0\|^2 &\leq \frac{1}{N}\sum_{k=1}^{N} 2\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 + \frac{1}{N}\sum_{k=1}^{N} 2\|\tilde{A}_d^k x_0\|^2 \\ &\leq 2\sum_{k=1}^{N}\frac{1}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 + 2\sup_k \|\tilde{A}_d^k x_0\|^2 \\ &\leq M_1\|x_0\|^2, \end{aligned}$$

   where we have used (4.5) and the power stability of $(\tilde{A}_d^n)_{n\geq 0}$. Similarly, we obtain the dual result. Hence by Lemma 2.21, we conclude that $(A_d^n)_{n\geq 0}$ is power stable.

2. For all $N \geq 1$ we have for all $x_0 \in X$,

$$\begin{aligned} \sum_{k=1}^{N}\frac{1}{k}\|A_d^k x_0\|^2 &\leq 2\sum_{k=1}^{N}\frac{1}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 + 2\sum_{k=1}^{N}\frac{1}{k}\|\tilde{A}_d^k x_0\|^2 \\ &\leq M_2\|x_0\|^2, \end{aligned} \tag{4.12}$$

   where we have used the finite Bergman distance and the power stability of $(\tilde{A}_d^n)_{n\geq 0}$.

   The power stability of $(\tilde{A}_d^n)_{n\geq 0}$ implies that $(\tilde{A}_d^n)_{n\geq 0}$ is bounded. By item 1., we have that $(A_d^n)_{n\geq 0}$ is bounded as well. Combining this with equation (4.12):

$$\begin{aligned} \ln(n+1)\|A_d^n x_0\|^2 &\leq \sum_{k=1}^{n}\frac{1}{k}\|A_d^n x_0\|^2 \\ &\leq \sum_{k=1}^{n}\frac{1}{k}\|A_d^{n-k}\|^2\|A_d^k x_0\|^2 \\ &\leq M_1\sum_{k=1}^{n}\frac{1}{k}\|A_d^k x_0\|^2 \leq M_1 M_2\|x_0\|^2. \end{aligned}$$

So we have that

$$\|A_d^n\|^2 \leq \frac{M_1 M_2}{\ln(n+1)}.$$

Since the right-hand side is less than one for sufficiently large n, by Lemma 2.13 we have that $(A_d^n)_{n\geq 0}$ is power stable.

3. Let x_0 be an element of X. Since $\sum_{k=1}^{\infty} \frac{1}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 < \infty$, for every $\varepsilon > 0$, there exists a n_ε such that

$$\sum_{k=n_\varepsilon}^{\infty} \frac{1}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 < \varepsilon. \tag{4.13}$$

Furthermore, the following inequality holds

$$\frac{1}{n}\sum_{k=1}^{n} \|A_d^k x_0\|^2 \leq \frac{1}{n}\sum_{k=1}^{n} 2\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 + \frac{1}{n}\sum_{k=1}^{n} 2\|\tilde{A}_d^k x_0\|^2$$

Using this inequality and Lemma 2.22, we have that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|A_d^k x_0\|^2 &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n_\varepsilon - 1} 2\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 \\ &\quad + \lim_{n\to\infty} \frac{1}{n}\sum_{k=n_\varepsilon}^{n} 2\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 + 0 \\ &\leq 0 + \lim_{n\to\infty} \sum_{k=n_\varepsilon}^{n} \frac{2}{k}\|A_d^k x_0 - \tilde{A}_d^k x_0\|^2 \leq 2\varepsilon. \end{aligned}$$

where we have used inequality (4.13).

Since this holds for all $\epsilon > 0$, we have shown that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|A_d^k x_0\|^2 = 0,$$

and so by Lemma 2.22 $(A_d^n)_{n\geq 0}$ is strongly stable.

This third item concludes the proof. □

4.4 Equivalence of the Bergman distances

In the previous sections we have derived properties of operators with finite Bergman distance. In this section, we show that the Cayley transform preserves Bergman distances.
We do the proof under the condition that the Hilbert space X is separable. However, Lemma 4.13 and Theorem 4.14 also holds for non-separable Hilbert spaces. Since separable Hilbert spaces are most often encountered in applications and it makes the proof easier to read, we use this condition. For non-separable Hilbert spaces the idea of the proof is the same as for separable Hilbert spaces, but we have to interchange the countable set of basis vectors for a uncountable one.
First, we define the inner product space H.

Definition 4.10 *Let H denote the space of Lebesgue measurable functions f from $[0,\infty)$ to the Hilbert space X such that:*

$$\int_0^\infty \|f(t)\|_X^2 t\,dt < \infty.$$

Note that this is a weighted $L^2([0,\infty],X)$-space On H we define the following inner product:

$$\langle f,g\rangle_H = \int_0^\infty \langle f(t),g(t)\rangle_X t\,dt. \tag{4.14}$$

The following result is easy to see.

Lemma 4.11 *The inner product space H defined in Definition 4.10 is a Hilbert space.*

To create an orthonormal basis for this Hilbert space, we use the generalized Laguerre polynomials $L_n^{(1)}(t)$ [34, p. 99]. These are defined by

$$L_{n-1}^{(1)}(2t) = \sum_{k=0}^{n-1} \binom{n}{n-k-1} \frac{(-2t)^k}{k!}, \qquad \text{for } n \geq 1 \text{ and } t \in [0,\infty). \tag{4.15}$$

Lemma 4.12 *Let H be the Hilbert space defined by Definition 4.10 and let $\{e_m\}_{m\in\mathbb{N}}$ be an orthonormal basis of X. The vectors $\varphi_{n,m}$ defined by:*

$$\varphi_{n,m}(t) = \frac{q_n(t)}{\sqrt{n}} e_m, \qquad n,m \geq 1, \tag{4.16}$$

with

$$q_n(t) = -2e^{-t} L_{n-1}^{(1)}(2t), \tag{4.17}$$

form an orthonormal basis in H.

Proof: We begin by showing that the sequence $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is orthonormal in H. Using equation (4.14), we find:

$$\begin{aligned}\langle \varphi_{n,m}, \varphi_{\nu,\mu}\rangle_H &= \int_0^\infty \langle \frac{-2e^{-t}L_{n-1}^{(1)}(2t)}{\sqrt{n}} e_m, \frac{-2e^{-t}L_{\nu-1}^{(1)}(2t)}{\sqrt{\nu}} e_\mu\rangle_X t\, dt \\ &= \frac{4}{\sqrt{n}\sqrt{\nu}} \int_0^\infty e^{-2t} t L_{n-1}^{(1)}(2t) L_{\nu-1}^{(1)}(2t) dt \, \langle e_m, e_\mu\rangle_X \\ &= \frac{1}{\sqrt{n}\sqrt{\nu}} \int_0^\infty e^{-\tau} \tau L_{n-1}^{(1)}(\tau) L_{\nu-1}^{(1)}(\tau) d\tau \, \langle e_m, e_\mu\rangle_X \\ &= \frac{1}{\sqrt{n}\sqrt{\nu}} \binom{n}{n-1} \delta_{(n-1)(\nu-1)} \delta_{m\mu} = \delta_{n\nu}\delta_{m\mu},\end{aligned}$$

where we use the orthogonality of the Laguerre polynomials in $L^2(0,\infty)$ with weight $\tau e^{-\tau}$, see [34, p. 99].
Next we show that the sequence $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is maximal in H. If $h \in H$ is orthogonal to every $\varphi_{n,m}$, then for all n and $m \geq 1$:

$$\langle \varphi_{n,m}, h\rangle_H = \int_0^\infty L_{n-1}^{(1)}(2t) \frac{-2e^{-t}}{\sqrt{n}} \langle e_m, h(t)\rangle_X t\, dt = 0.$$

From the maximality of $\{L_{n-1}^{(1)}(2t)e^{-t}t\}_{n\geq 1}$ in $L^2(0,\infty)$, see [34, p. 107], we conclude that for all $m \geq 1$,

$$\langle e_m, h(t)\rangle_X = 0 \quad \text{almost everywhere.}$$

This, combined with the maximality of $\{e_m\}_{m\in\mathbb{N}}$ in X, leads to the conclusion that the Lebesgue measurable function $h(t) = 0$ almost everywhere. So $h = 0$ in H and $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is maximal. □

Lemma 4.12 gives us the following Parseval equality:

$$\|f\|_H^2 = \sum_{n=1}^\infty \sum_{m=1}^\infty |\langle f, \varphi_{n,m}\rangle_H|^2. \tag{4.18}$$

We use the Laguerre polynomials to write the Cayley transform as an integral.

Lemma 4.13 *Let q_n be defined by equation (4.17), let A generate a C_0-semigroup with growth bound $\omega_0 < 1$, and let A_d be the Cayley transform of A. Then,*

$$\int_0^\infty q_n(t) e^{At} x_0 dt = A_d^n x_0 - x_0, \qquad x_0 \in X. \tag{4.19}$$

Proof: Using equation (4.17) and (4.15) we rewrite $q_n(t)$ as follows:

$$\begin{aligned} q_n(t) &= -2e^{-t} L_{n-1}^{(1)}(2t) \\ &= -2e^{-t} \sum_{k=0}^{n-1} \binom{n}{n-k-1} \frac{(-2t)^k}{k!} \\ &= 2 \sum_{k=0}^{n-1} \frac{n!}{(n-k-1)!(k+1)!k!} (-1)^{k+1} (2t)^k e^{-t} \\ &= 2 \sum_{\ell=1}^{n} \frac{n!}{(n-\ell)!\ell!(\ell-1)!} (-1)^{\ell} (2t)^{\ell-1} e^{-t} \\ &= \sum_{\ell=1}^{n} \binom{n}{\ell} (-2)^{\ell} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t}, \end{aligned}$$

where we introduced $\ell = k+1$ in the fourth equality.
We insert this expression into the left-hand side of equation (4.19) and use

$$(A-I)^{-\ell} x_0 = (-1)^{\ell} R(1,A)^{-\ell} x_0 = \int_0^\infty (-1)^{\ell} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t} e^{At} x_0 dt,$$

see [16, p. 57].

$$\begin{aligned} \int_0^\infty q_n(t) e^{At} x_0 dt &= \sum_{\ell=1}^{n} \binom{n}{\ell} \int_0^\infty (-2)^{\ell} \frac{t^{\ell-1}}{(\ell-1)!} e^{-t} e^{At} x_0 dt \\ &= \sum_{\ell=1}^{n} \binom{n}{\ell} (2)^{\ell} (A-I)^{-\ell} x_0 \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} (2)^{\ell} (A-I)^{-\ell} x_0 - x_0 \\ &= \left(I + 2(A-I)^{-1}\right)^n x_0 - x_0 \\ &= A_d^n x_0 - x_0, \end{aligned}$$

where we used equation (1.17). Thus equation (4.19) holds. □

The previous lemma's enable us to prove the main result of this section. Namely, that the Cayley transform preserves the Bergman distances.

Theorem 4.14 *Let A and $\tilde{A}$ generate a C_0-semigroup, with growth bounds $\omega_0 < 1$ and $\tilde{\omega}_0 < 1$ and let A_d and $\tilde{A}_d$ be their Cayley transforms. Then $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have finite Bergman distance if and only if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have finite Bergman distance.*

Furthermore, for all $x_0 \in X$

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|_X^2 \frac{1}{t} dt = \sum_{n=1}^\infty \frac{1}{n} \|(A_d^n - \tilde{A}_d^n)x_0\|_X^2. \tag{4.20}$$

Proof: First, we write the left-hand side of (4.20) in the norm on H, see Definition 4.10. Next, we apply the Parseval identity of H, see equation (4.18):

$$\begin{aligned}
\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|_X^2 \frac{1}{t} dt &= \int_0^\infty \left\| \frac{(e^{At} - e^{\tilde{A}t})x_0}{t} \right\|_X^2 t dt \\
&= \left\| \frac{(e^{At} - e^{\tilde{A}t})x_0}{t} \right\|_H^2 \\
&= \sum_{n=1}^\infty \sum_{m=1}^\infty \left| \left\langle \frac{(e^{At} - e^{\tilde{A}t})x_0}{t}, \varphi_{n,m} \right\rangle_H \right|^2.
\end{aligned}$$

Zooming in on the inner product, and applying equation (4.16) and Lemma 4.13, we find

$$\begin{aligned}
\left\langle \frac{(e^{At} - e^{\tilde{A}t})x_0}{t}, \varphi_{n,m} \right\rangle_H &= \int_0^\infty \left\langle \frac{(e^{At} - e^{\tilde{A}t})x_0}{t}, \frac{q_n(t)}{\sqrt{n}} e_m \right\rangle_X t dt \\
&= \frac{1}{\sqrt{n}} \int_0^\infty \left\langle q_n(t)(e^{At} - e^{\tilde{A}t})x_0, e_m \right\rangle_X dt \\
&= \frac{1}{\sqrt{n}} \left\langle \int_0^\infty q_n(t)(e^{At} - e^{\tilde{A}t})x_0 dt, e_m \right\rangle_X \\
&= \frac{1}{\sqrt{n}} \left\langle (A_d^n - \tilde{A}_d^n)x_0, e_m \right\rangle_X.
\end{aligned}$$

We zoom out again and use the Parseval equation of X for the orthonormal basis $\{e_m\}_{m\in\mathbb{N}}$.

$$\begin{aligned}
\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|_X^2 \frac{1}{t} dt &= \sum_{n=1}^\infty \sum_{m=1}^\infty \left| \frac{1}{\sqrt{n}} \left\langle (A_d^n - \tilde{A}_d^n)x_0, e_m \right\rangle_X \right|^2 \\
&= \sum_{n=1}^\infty \frac{1}{n} \sum_{m=1}^\infty \left| \left\langle (A_d^n - \tilde{A}_d^n)x_0, e_m \right\rangle_X \right|^2 \\
&= \sum_{n=1}^\infty \frac{1}{n} \left\| (A_d^n - \tilde{A}_d^n)x_0 \right\|_X^2.
\end{aligned}$$

Thus equation (4.20) holds.

In particular, this shows that $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance if and only if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have. □

4.5 Proof of Theorem 4.7

In this section, we return to Theorem 4.7. With the results from the Sections 4.2, 4.3, and 4.4, we are able to prove it. First, we extend the formulation of Theorem 4.7.

Theorem 4.15 *Let $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ be C_0-semigroups on the Hilbert space X with growth bounds $\omega_0 < 1$ and $\tilde{\omega}_0 < 1$ and let the Cayley transforms of A and $\tilde{A}$ exist and be denoted by A_d and $\tilde{A}_d$, respectively.*

A *If $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance, then the following three equivalences holds.*

1. *$(A_d^n)_{n\geq 0}$ is bounded if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is bounded.*
2. *$(A_d^n)_{n\geq 0}$ is power stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is power stable.*
3. *$(A_d^n)_{n\geq 0}$ is strongly stable if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is strongly stable.*

B *If $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have a finite Bergman distance, then the following three equivalences holds.*

1. *$(e^{At})_{t\geq 0}$ is bounded if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is bounded.*
2. *$(e^{At})_{t\geq 0}$ is exponentially stable if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is exponentially stable.*
3. *$(e^{At})_{t\geq 0}$ is strongly stable if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is strongly stable.*

Proof: Recalling from Theorem 4.14, that $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance if and only if $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$ have a finite Bergman distance.
So to prove item *A1* the argument goes as follows. The finite Bergman distance of $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ implies the finite Bergman distance between $(A_d^n)_{n\geq 0}$ and $(\tilde{A}_d^n)_{n\geq 0}$. Using the first item of Theorem 4.9, we conclude that $(A_d^n)_{n\geq 0}$ is bounded if and only if $(\tilde{A}_d^n)_{n\geq 0}$ is bounded.
Items *A2* and *A3* are proved similarly. Only now we use the second and third item of Theorem 4.9 instead of the first item.
The equivalences *B1*, *B2* and *B3* are proved in same as the other three items. Only now we use Theorem 4.8 instead of Theorem 4.9. □

Now, we return to the central question in this thesis: If we know that the semigroup $(e^{At})_{t\geq 0}$ is strongly stable, what can be said about $(A_d^n)_{n\geq 0}$? Or what can be said about sequences $(\tilde{A}_d^n)_{n\geq 0}$ at a finite Bergman distance of $(A_d^n)_{n\geq 0}$.
Before answering this question, we first recall the following result by Guo and Zwart [21, Theorem 4.3].

Lemma 4.16 *Let $(e^{At})_{t\geq 0}$ be a C_0-semigroup and let the Cayley transform of A, denoted by A_d, exist. If $(e^{At})_{t\geq 0}$ and $(A_d^n)_{n\geq 0}$ are bounded, and $(e^{At})_{t\geq 0}$ is strongly stable, then $(A_d^n)_{n\geq 0}$ is strongly stable.*

Hence, if we combine this lemma with Theorem 4.15, we find that if $(e^{At})_{t\geq 0}$ and $(A_d^n)_{n\geq 0}$ are bounded, then the strong stability of $(e^{At})_{t\geq 0}$ implies the strong stability of $(e^{\tilde{A}t})_{t\geq 0}$, $(A_d^n)_{n\geq 0}$, and $(\tilde{A}_d^n)_{n\geq 0}$, provided the two semigroups or the two discrete operators have finite Bergman distance.
The above argument can be used to extend the class of infinitesimal generators with a power bounded cogenerator. If we combine by Theorems 1.9 and 4.15, we obtain the following corollary.

Corollary 4.17 *Let A be the infinitesimal generator of a contraction semigroup on X, and let $\tilde{A}$ be another generator with finite Bergman distance to A. Then $(\tilde{A}_d^n)_{n\geq 0}$ is bounded.*

This motivates the question which semigroups or cogenerators have finite Bergman distance. In the next section we derive some sufficient conditions.

4.6 Applications

In this section we present some classes of semigroups with a bounded Bergman distance. We begin with bounded perturbations of exponentially stable semigroups.

Lemma 4.18 *Let A and $\tilde{A}$ generate exponentially stable semigroups and let $A - \tilde{A}$ be bounded, then $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance.*

Proof: Since the semigroups are exponentially stable, there exist M_1, M_2, ω_1 and ω_2 all positive such that $\|e^{At}\| \leq M_1 e^{-\omega_1 t}$ and $\|e^{\tilde{A}t}\| \leq M_2 e^{-\omega_2 t}$, respectively. We show that these semigroups satisfy equation (4.1) by cutting the time interval $[0, \infty)$ into two parts, and showing, for each part, that the integral is finite.

The first time interval is from 0 to 1. Using the perturbation formula $e^{At}x_0 = e^{\tilde{A}t}x_0 + \int_0^t e^{As}(A-\tilde{A})e^{\tilde{A}(t-s)}x_0 ds$, we find

$$\begin{aligned}
\int_0^1 \left\| \left(e^{At} - e^{\tilde{A}t}\right) x_0 \right\|^2 \frac{1}{t} dt &= \int_0^1 \left\| \int_0^t e^{As}(A-\tilde{A})e^{\tilde{A}(t-s)}x_0 ds \right\|^2 \frac{1}{t} dt \\
&\leq \int_0^1 \left(\int_0^t \|e^{As}(A-\tilde{A})e^{\tilde{A}(t-s)}x_0\| ds \right)^2 \frac{1}{t} dt \\
&\leq \int_0^1 \left(\int_0^t M_1 \|A-\tilde{A}\| M_2 \|x_0\| ds \right)^2 \frac{1}{t} dt \\
&\leq \int_0^1 \left(t M_1 M_2 \|A-\tilde{A}\| \|x_0\| \right)^2 \frac{1}{t} dt \\
&\leq M_1^2 M_2^2 \|A-\tilde{A}\|^2 \|x_0\|^2 \int_0^1 t dt < \infty. \qquad (4.21)
\end{aligned}$$

On the second time interval $(1, \infty)$, we use the exponential stability

$$\begin{aligned}
\int_1^\infty \left\| \left(e^{At} - e^{\tilde{A}t}\right) x_0 \right\|^2 \frac{1}{t} dt &\leq \int_1^\infty \left\| \left(e^{At} - e^{\tilde{A}t}\right) x_0 \right\|^2 dt \\
&\leq \int_1^\infty 2\|e^{At}x_0\|^2 + 2\|e^{\tilde{A}t}x_0\|^2 dt \\
&\leq \frac{M_1^2}{\omega_1} e^{-2\omega_1} \|x_0\|^2 + \frac{M_2^2}{\omega_2} e^{-2\omega_2} \|x_0\|^2 < \infty. \qquad (4.22)
\end{aligned}$$

Since (4.21) and (4.22) hold for all $x_0 \in X$, equation (4.1) holds.
The proof for the adjoint operators goes similarly, and hence we conclude the proof. □

We remark that combining this lemma with Theorem 4.8 gives the following corollary.

Corollary 4.19 *Let A generate an exponentially stable semigroup, let Q be a bounded operator, and define $\tilde{A}$ as $\tilde{A} = A + Q$. Then $(e^{At})_{t\geq 0}$ and $(e^{\tilde{A}t})_{t\geq 0}$ have a finite Bergman distance if and only if $(e^{\tilde{A}t})_{t\geq 0}$ is exponentially stable.*

We apply the previous result to the closed loop generator in the linear quadratic optimal control problem.

Lemma 4.20 *Let A generate an exponentially stable contraction semigroup, and let B be bounded. By Π we denote the stabilizing solution of the algebraic Riccati equation, corresponding to the optimal control problem*

$$\min_u \int_0^\infty \|Cx(t)\|^2 + \langle u(t), Ru(t) \rangle dt,$$

where $C \in \mathcal{L}(X,Y)$ and $R \in \mathcal{L}(U)$ with $R \geq r_0 I > 0$ and U and Y are Hilbert spaces, see [10, Chapter 6]. Then the Cayley transform of $A - BR^{-1}B^*\Pi$ is strongly stable.

Proof: By Lemma 4.18 the semigroups $(e^{At})_{t\geq 0}$ and $(e^{(A-BR^{-1}B^*\Pi)t})_{t\geq 0}$ have a finite Bergman distance. Since $(e^{At})_{t\geq 0}$ is a contraction semigroup, we know that $(A_d^n)_{n\geq 0}$ is bounded by one, see Theorem 1.9. It is strongly stable a well, since $(e^{At})_{t\geq 0}$ is exponentially stable, see Lemma 4.16. Theorem 4.15 proves the assertion. □

Lemma 4.18 provides an answer for bounded perturbations of A. Next we study a class of unbounded perturbations. We show that a subset of the class of Desch-Schappacher perturbations leads to pairs of semigroups with finite Bergman distance. First, we introduce the class of Desch-Schappacher perturbations, see Engel and Nagel [16, Section III.3.a].
We need some notation. By X_{-1} we denote the dual of the domain of A^*. We regard X as the pivot space, and thus we have the inclusions $D(A) \subset X \subset X_{-1}$ all with dense injection. Next we define $A_{-1} : X \to X_{-1}$ as follows

$$\langle A_{-1}x_0, x_* \rangle_{X_{-1}\times D(A*)} = \langle x_0, A^* x_* \rangle, \tag{4.23}$$

with $x_0 \in X$, $x_* \in D(A^*)$, and $\langle \cdot, \cdot \rangle_{X_{-1}\times D(A*)}$ denotes the duality product of $D(A^*)$ and its dual.
For $x_0 \in D(A)$, we see from (4.23) that $A_{-1}x_0 = Ax_0$, and so A_{-1} is an extension of A. It can be shown that A_{-1} generates a C_0-semigroup on X_{-1}, see [35] and [16] for this and more results on X_{-1} and A_{-1}.
We continue by defining $\mathcal{X}_{t_0}$ as the space of all strongly continuous, $\mathcal{L}(X)$-valued functions,

$$\mathcal{X}_{t_0} = C\left([0,t_0], \mathcal{L}(X)\right), \qquad \text{with the norm } \|F\|_\infty = \sup_{r\in[0,t_0]} \|F(r)\|_{\mathcal{L}(X)}.$$

Note that $\mathcal{X}_{t_0}$ is a Banach space. For the C_0-semigroup $(e^{At})_{t\geq 0}$ and the operator $B \in \mathcal{L}(X, X_{-1})$ from X to the extrapolation space X_{-1} we define the abstract Volterra operator V_B on the space $\mathcal{X}_{t_0}$ by

$$(V_B F)(t) = \int_0^t e^{A_{-1}(t-r)} BF(r)dr, \qquad \text{for all } t \in [0,t_0] \text{ and } F \in \mathcal{X}_{t_0}.$$

Note that we use the extended semigroup on X_{-1} in this definition. The class of Desch-Schappacher perturbations is defined by

$$\mathcal{S}_{t_0}^{DS} = \{B \in \mathcal{L}(X, X_{-1}) \mid V_B \in \mathcal{L}(\mathcal{X}_{t_0}),\ \|V_B\| < 1\} \tag{4.24}$$

In [16, Theorem III.3.1] it is shown that for $B \in \mathcal{S}_{t_0}^{DS}$ the operator $(A_{-1} + B)_X$ defined by

$$\begin{aligned}(A_{-1} + B)_X x_0 &= A_{-1}x_0 + Bx_0,\\ D((A_{-1} + B)_X) &= \{x_0 \in X \mid A_{-1}x_0 + Bx_0 \in X\},\end{aligned}$$

generates a C_0-semigroup on X.
Furthermore, if we restrict the class of Desch-Schappacher perturbations by two extra conditions, then a perturbation B in this restricted class leads to a finite Bergman distance.

Lemma 4.21 *Let A generate an exponentially stable semigroup and let $B \in \mathcal{S}_{t_0}^{DS}$. If, for some $M > 1$ and $\alpha > 0$*

$$\|(V_B)\|_{\mathcal{L}(\mathcal{X}_t)} \leq Mt^{\alpha}, \quad \text{for } t \in (0, t_0),$$

and, for some $q \in (0, 1)$

$$\|R(\lambda, A_{-1})B\| \leq q, \quad \text{for all } \lambda \in \mathbb{C}^+ = \{s \in \mathbb{C} \mid \mathrm{Re}(s) > 0\}, \tag{4.25}$$

then the semigroups generated by A and $(A_{-1} + B)_X$ have a finite Bergman distance.

Proof: First, we define $\tilde{A} = (A_{-1} + B)_X$. It follows from equation (4.25), that the semigroup generated by $\tilde{A}$ is exponentially stable, see [29, Proposition 5.8].
Now, the proof is similar to the proof of Lemma 4.20. Let M_1, M_2, ω_1 and ω_2 be positive constants such that $\|e^{At}\| \leq M_1 e^{-\omega_1 t}$ and $\|e^{\tilde{A}t}\| \leq M_2 e^{-\omega_2 t}$, respectively. We show that these semigroups satisfy equation (4.1) by cutting the time interval $[0, \infty)$ into two parts, and showing, for each part, that the integral is finite.
The first time interval is from 0 to t_0. We use the variation of constant formula.

$$\begin{aligned}\int_0^{t_0} \left\| \left(e^{At} - e^{\tilde{A}t}\right) x_0 \right\|^2 \frac{1}{t} dt &= \int_0^{t_0} \left\| \int_0^t e^{A_{-1}(t-s)} (A - \tilde{A}) e^{\tilde{A}s} x_0 ds \right\|^2 \frac{1}{t} dt\\ &= \int_0^{t_0} \left\| (V_B e^{\tilde{A}\cdot})(t) x_0 \right\|^2 \frac{1}{t} dt\\ &\leq \int_0^{t_0} M^2 t^{2\alpha - 1} M_2^2 \|x_0\|^2 dt\\ &= \frac{M^2 M_2^2}{2\alpha} t_0^{2\alpha} \|x_0\|^2 < \infty.\end{aligned} \tag{4.26}$$

This holds for all $x_0 \in X$.
The second time interval is from t_0 to ∞. The proof for this interval is similar to the second part of the proof of Lemma 4.18, and is therefor omitted.
For the adjoint operators we make the following observation:

$$\begin{aligned}
\left\| \left(e^{\tilde{A}^* t} - e^{A^* t} \right) x_0 \right\| &= \sup_{\|y_0\|=1} |\langle y_0, (e^{\tilde{A}^* t} - e^{A^* t}) x_0 \rangle| \\
&= \sup_{\|y_0\|=1} |\langle (e^{\tilde{A} t} - e^{A t}) y_0, x_0 \rangle| \\
&= \sup_{\|y_0\|=1} |\langle \int_0^t e^{A(t-s)} (A - \tilde{A}) e^{\tilde{A} s} ds \, y_0, x_0 \rangle| \\
&= \sup_{\|y_0\|=1} |\langle V_B (e^{\tilde{A} \cdot}) y_0, x_0 \rangle| \\
&\leq \sup_{\|y_0\|=1} M t^{\alpha} M_2 \|y_0\| \|x_0\|. \\
&= M t^{\alpha} M_2 \|x_0\|.
\end{aligned}$$

Using this inequality, we find similar to (4.26), that

$$\int_0^{t_0} \| \left(e^{A^* t} - e^{\tilde{A}^* t} \right) x_0 \|^2 \frac{1}{t} dt \leq \frac{M^2 M_2^2}{2\alpha} t_0^{2\alpha} \|x_0\|^2.$$

Concluding, we see that the semigroups generated by A and $(A_{-1} + B)_X$ have a finite Bergman distance. □

4.7 Conclusions

In this chapter we introduced the Bergman distance for semigroups. This distance is the integrated difference between the semigroups with the weight $\frac{1}{t}$. For cogenerators we introduced the Bergman distance as well. In this case the distance is an infinite sum of the difference of the powers with weight $\frac{1}{n}$
If two semigroups have a finite Bergman distance, then they have the same stability properties. So if one semigroup is stable or bounded so is the other one. For two cogenerators with a finite Bergman distance holds the same.
Another property of the Bergman distance is, that it is preserved by the Cayley transform. This means that if two semigroups have a finite Bergman distance, then their cogenerators have a finite Bergman distance as well.
The main theorem of this chapter combines these properties. For two semigroups with a finite Bergman distance, their cogenerators have the same

stability properties. And also two semigroups corresponding to cogenerators with a finite Bergman distance, share the same stability properties.
Furthermore, we gave some examples of generators with finite Bergman distance and we applied this to the quadratic optimal control problem.
In Chapter 5 examine the equivalence classes created by the Bergman distance further and prove other shared properties.

Chapter 5

Extension Bergman distance

In Chapter 4 we introduced the Bergman distance. In this chapter we show that the Bergman distance divides semigroups and cogenerators into equivalence classes. On an equivalence class the Bergman distance defines a metric. We derive some properties of semigroups within one equivalence class and we characterize the classes of finite-dimensional semigroups and cogenerators.

5.1 Equivalence classes

The finite Bergman distance is defined in Definition 4.1 and Definition 4.4 for semigroups and cogenerators, respectively. In the next subsections we show that the finite Bergman distance divides the semigroups and the co-generators into equivalence classes.

5.1.1 Classes of semigroups

The Bergman distance on the Hilbert space X can be seen as a binary relation on the set of operators on X which generate a C_0-semigroup. We define $A \overset{B}{\sim} \tilde{A}$ if the semigroups generates by A and $\tilde{A}$ have a finite Bergman distance, that is if the following two equations hold.

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt < \infty,$$
$$\int_0^\infty \|(e^{A^*t} - e^{\tilde{A}^*t})x_0\|^2 \frac{1}{t} dt < \infty.$$

These equations are the same as equations (4.1) and (4.2).

Lemma 5.1 *The binary relation* $\overset{B}{\sim}$ *on the set of operators on* X *which generate a semigroup, is an equivalence relation.*

Proof: We have to check the reflexivity, symmetry and transitivity of $\overset{B}{\sim}$. The reflexivity, $A \overset{B}{\sim} A$, and the symmetry, if $A \overset{B}{\sim} \tilde{A}$ then $\tilde{A} \overset{B}{\sim} A$, are trivial. So it remains to show transitivity. Using the inequality

$$\|x - z\|^2 \le 2\|x - y\|^2 + 2\|y - z\|^2,$$

the transitivity, if $A \overset{B}{\sim} B$ and $B \overset{B}{\sim} C$ then $A \overset{B}{\sim} C$, is easy to see from

$$\int_0^\infty \|(e^{At} - e^{Ct})x_0\|^2 \frac{1}{t} dt \le \int_0^\infty \left(2\|(e^{At} - e^{Bt})x_0\|^2 + 2\|(e^{Bt} - e^{Ct})x_0\|^2\right) \frac{1}{t} dt$$
$$= 2\int_0^\infty \|(e^{At} - e^{Bt})x_0\|^2 \frac{1}{t} dt + 2\int_0^\infty \|(e^{Bt} - e^{Ct})x_0\|^2 \frac{1}{t} dt < \infty.$$

And the same holds for the adjoint operators $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$.
So the binary relation $\overset{B}{\sim}$ is an equivalence relation. □

The equivalence relation $\overset{B}{\sim}$ divides the set of infinitesimal generators on X into equivalence classes. Such a class is denoted by $[A]_B$, and defined by,

$$[A]_B = \{\tilde{A} \mid A \overset{B}{\sim} \tilde{A}\}. \tag{5.1}$$

Note that the choice of the representative of the class is arbitrary.
Every two semigroups in an equivalence class have a finite Bergman distance by definition. On an equivalence class, the Bergman distance defines a metric.

Lemma 5.2 *The Bergman distance* $d\left(e^{At}, e^{\tilde{A}t}\right)$ *as defined in Definition 4.3 defines a metric on the equivalence classes.*

Proof: We need to check the four properties of a metric.

M1 d is real-valued, finite and nonnegative. From Definition 4.3 of the Bergman distance, it is easy to see that d is real-valued and nonnegative. The definition of the equivalence classes provides that d is finite.

M2 $d(x,y) = 0$ if and only if $x = y$. If $d\left(e^{At}, e^{\tilde{A}t}\right) = 0$ then for all x_0

$$\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt = 0.$$

Thus $(e^{At} - e^{\tilde{A}t})x_0 = 0$ almost everywhere. Since the semigroups are strongly continuous, this means $e^{At}x_0 = e^{\tilde{A}t}x_0$ for all $t \geq 0$ and for all $x_0 \in X$. Hence $(e^{At})_{t\geq 0} = (e^{\tilde{A}t})_{t\geq 0}$.

The other implication is trivial.

M3 $d(x,y) = d(y,x)$. The symmetry trivially follows from Definition 4.3.

M4 $d(x,y) \leq d(x,z) + d(z,y)$. We know from Remark 4.2 and Definition 4.3 that for all $x_0 \in X$,

$$d\left(e^{At}, e^{Bt}\right) \|x_0\| \geq \sqrt{\int_0^\infty \|(e^{At} - e^{Bt})x_0\|^2 \frac{1}{t} dt},$$

$$d\left(e^{Bt}, e^{Ct}\right) \|x_0\| \geq \sqrt{\int_0^\infty \|(e^{Bt} - e^{Ct})x_0\|^2 \frac{1}{t} dt}.$$

Now we take the square of their sum and use the Cauchy-Schwarz inequality in the second step.

$$\begin{aligned}
&\left(d\left(e^{At}, e^{Bt}\right) + d\left(e^{Bt}, e^{Ct}\right)\right)^2 \|x_0\|^2 \\
&\geq \int_0^\infty \|(e^{At} - e^{Bt})x_0\|^2 \frac{1}{t} dt + \int_0^\infty \|(e^{Bt} - e^{Ct})x_0\|^2 \frac{1}{t} dt \\
&\quad + 2\sqrt{\int_0^\infty \|(e^{At} - e^{Bt})x_0\|^2 \frac{1}{t} dt}\sqrt{\int_0^\infty \|(e^{Bt} - e^{Ct})x_0\|^2 \frac{1}{t} dt} \\
&\geq \int_0^\infty \|(e^{At} - e^{Bt})x_0\|^2 \frac{1}{t} dt + \int_0^\infty \|(e^{Bt} - e^{Ct})x_0\|^2 \frac{1}{t} dt \\
&\quad + 2\int_0^\infty \|(e^{At} - e^{Bt})x_0\| \|(e^{Bt} - e^{Ct})x_0\| \frac{1}{t} dt \\
&= \int_0^\infty \left(\|(e^{At} - e^{Bt})x_0\| + \|(e^{Bt} - e^{Ct})x_0\|\right)^2 \frac{1}{t} dt \\
&\geq \int_0^\infty \|(e^{At} - e^{Ct})x_0\|^2 \frac{1}{t} dt, \qquad (5.2)
\end{aligned}$$

where we used the triangle inequality for the norm of X in the last step. Since inequality (5.2) holds for all $x_0 \in X$, we have

$$d\left(e^{At}, e^{Ct}\right)^2 \leq \left(d\left(e^{At}, e^{Bt}\right) + d\left(e^{Bt}, e^{Ct}\right)\right)^2.$$

Hence the triangle inequality holds.

The Bergman distance defines a metric. □

5.1.2 Classes of cogenerators

The Bergman distance on the Hilbert space X can be seen as a binary relation on the set of cogenerators on X. We define $A_d \overset{B}{\sim} \tilde{A}_d$ if the cogenerators A and $\tilde{A}$ have a finite Bergman distance. that is if the following two equations hold.

$$\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - \tilde{A}_d^k)x_0\|^2 < \infty,$$
$$\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^{*k} - \tilde{A}_d^{*k})x_0\|^2 < \infty.$$

These equations are the same as equations (4.5) and (4.6).

Lemma 5.3 *The binary relation $\overset{B}{\sim}$ on the set of cogenerators on X, is an equivalence relation.*

Proof: We have to check the reflexivity, symmetry and transitivity of $\overset{B}{\sim}$. The reflexivity, $A_d \overset{B}{\sim} A_d$, and the symmetry, if $A_d \overset{B}{\sim} \tilde{A}_d$ then $\tilde{A}_d \overset{B}{\sim} A_d$, are trivial. So it remains to show transitivity. Using the inequality

$$\|x - z\|^2 \leq 2\|x - y\|^2 + 2\|y - z\|^2,$$

the transitivity, if $A_d \overset{B}{\sim} B_d$ and $B_d \overset{B}{\sim} C_d$ then $A_d \overset{B}{\sim} C_d$, is easy to see from

$$\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - C_d^k)x_0\|^2 \leq \sum_{k=1}^{\infty} \frac{1}{k} \left(2\|(A_d^k - B_d^k)x_0\|^2 + 2\|(B_d^k - C_d^k)x_0\|^2\right)$$
$$= 2\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - B_d^k)x_0\|^2 + 2\sum_{k=1}^{\infty} \frac{1}{k} \|(B_d^k - C_d^k)x_0\|^2 < \infty.$$

And the same holds for the adjoint operators $\tilde{A}_d$, $\tilde{B}_d$, and $\tilde{C}_d$.

So the binary relation $\overset{B}{\sim}$ is an equivalence relation. □

The equivalence relation $\overset{B}{\sim}$ divides the set of cogenerators on X into equivalence classes. Such a class is denoted by $[A_d]_B$, and defined by,

$$[A_d]_B = \{\tilde{A}_d \mid A_d \overset{B}{\sim} \tilde{A}_d\}. \tag{5.3}$$

Note that the choice of the representative of the class is arbitrary.
Every two cogenerators in an equivalence class have a finite Bergman distance by definition. On an equivalence class, the Bergman distance defines a metric.

Lemma 5.4 *The Bergman distance* $d\left(A_d^n, \tilde{A}_d^n\right)$ *as defined in Definition 4.6 defines a metric on the equivalence classes.*

Proof: We need to check the four properties of a metric.

M1 d is real-valued, finite and nonnegative. From Definition 4.6 of the Bergman distance, it is easy to see that d is real-valued and nonnegative. The definition of the equivalence classes provides that d is finite.

M2 $d(x, y) = 0$ if and only if $x = y$. If $d\left(A_d^n, \tilde{A}_d^n\right) = 0$ then for all x_0

$$\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - \tilde{A}_d^k)x_0\|^2 = 0.$$

In particular, $(A_d - \tilde{A}_d)x_0 = 0$ for all $x_0 \in X$. Hence $A_d = \tilde{A}_d$.

The implication the other way is trivial.

M3 $d(x, y) = d(y, x)$. The symmetry trivially follows from Definition 4.6.

M4 $d(x, y) \leq d(x, z) + d(z, y)$. We know from Remark 4.5 and Definition 4.6 that for all $x_0 \in X$,

$$d\left(A_d^n, B_d^n\right)\|x_0\| \geq \sqrt{\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - B_d^k)x_0\|^2},$$

$$d\left(B_d^n, C_d^n\right)\|x_0\| \geq \sqrt{\sum_{k=1}^{\infty} \frac{1}{k} \|(B_d^k - C_d^k)x_0\|^2}.$$

Now we take the square of their sum and use the Cauchy-Schwarz inequality in the second step.

$$
\begin{aligned}
&\left(d\left(A_d^n, B_d^n\right) + d\left(B_d^n, C_d^n\right)\right)^2 \|x_0\|^2 \\
&\quad \geq \sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - B_d^k)x_0\|^2 + \sum_{k=1}^{\infty} \frac{1}{k} \|(B_d^k - C_d^k)x_0\|^2 \\
&\qquad + 2\sqrt{\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - B_d^k)x_0\|^2}\sqrt{\sum_{k=1}^{\infty} \frac{1}{k} \|(B_d^k - C_d^k)x_0\|^2} \\
&\quad \geq \sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - B_d^k)x_0\|^2 + \sum_{k=1}^{\infty} \frac{1}{k} \|(B_d^k - C_d^k)x_0\|^2 \\
&\qquad + 2\sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - B_d^k)x_0\| \|(B_d^k - C_d^k)x_0\| \\
&\quad \geq \sum_{k=1}^{\infty} \frac{1}{k} \left(\|(A_d^k - B_d^k)x_0\| + \|(B_d^k - C_d^k)x_0\|\right)^2 \\
&\quad \geq \sum_{k=1}^{\infty} \frac{1}{k} \|(A_d^k - C_d^k)x_0\|^2,
\end{aligned} \tag{5.4}
$$

where we used the triangle inequality for the norm of X in the last step. Since inequality (5.4) holds for all $x_0 \in X$, we have

$$
d\left(e^{At}, e^{Ct}\right)^2 \leq \left(d\left(e^{At}, e^{Bt}\right) + d\left(e^{Bt}, e^{Ct}\right)\right)^2.
$$

Hence the triangle inequality holds.

The Bergman distance defines a metric. □

5.2 Class properties

In this section we study the elements belonging to one equivalence class. First we show that if A generates a C_0-group then the other elements in its class also generate a C_0-group. Furthermore, we define stable states and show that these stable state are the same for the generators in an equivalence class.

Lemma 5.5 *Let A and $\tilde{A}$ generate a C_0-semigroup and $A \overset{B}{\sim} \tilde{A}$. If A generates a C_0-group on X, then also $\tilde{A}$ generates a C_0-group on X.*

Proof: The operator $\tilde{A}$ generates a C_0-group if there exists a $t_0 > 0$ such that $e^{\tilde{A}t_0}$ is invertible, that is $\mathrm{Ker}(e^{\tilde{A}t_0}) = \{0\}$ and $\mathrm{Ran}(e^{\tilde{A}t_0}) = X$, see [23, Theorem 2.6-10].
First, we show that if A generates a C_0-group and equation (4.1) is satisfied, then there exists a small $t_1 > 0$ such that $\mathrm{Ker}(e^{\tilde{A}t}) = \{0\}$ for $t \in (0, t_1)$.
Since $\left(e^{\tilde{A}t}\right)_{t\geq 0}$ is a semigroup and since $\left(e^{At}\right)_{t\in\mathbb{R}}$ is a group, there exists a $M \geq 1$ such that for all $t \in [0, 1]$ and all $x \in X$,

$$\|e^{\tilde{A}t}\| \leq M, \quad \|e^{At}\| \leq M, \text{ and } \|e^{At}x\| \geq \frac{1}{M}\|x\|. \tag{5.5}$$

Assume that there exist a $t_1 \in (0, 1)$ and a $x_0 \in X$ such that

$$\|e^{\tilde{A}t_1}x_0\| < \frac{1}{2M^2}\|x_0\|. \tag{5.6}$$

Using equation (5.5) and the semigroup property, we have that $\|e^{\tilde{A}t}x_0\| < \frac{1}{2M}\|x_0\|$ for all $t \in [t_1, 1]$. With equations (5.5) and (5.6), we can estimate the following integral,

$$\int_{t_1}^{1} \|(e^{At} - e^{\tilde{A}t})x_0\|^2 \frac{1}{t} dt \geq \int_{t_1}^{1} \left(\|e^{At}x_0\| - \|e^{\tilde{A}t}x_0\|\right)^2 \frac{1}{t} dt \tag{5.7}$$

$$\geq \int_{t_1}^{1} (\frac{1}{M} - \frac{1}{2M})^2 \|x_0\|^2 \frac{1}{t} dt \tag{5.8}$$

$$\geq \frac{-\ln t_1}{(2M)^2}\|x_0\|^2. \tag{5.9}$$

Note that $-\ln t_1$ is positive. Since equation (4.3) holds, assumption (5.6) cannot hold for sufficiently small $t_1 > 0$. So there exists a $t_1 > 0$ and a $\delta > 0$ such that for all $x \in X$ and $t \in [0, t_0]$

$$\|e^{\tilde{A}t}x\| \geq \delta\|x\|. \tag{5.10}$$

In particular this implies that $\mathrm{Ker}(e^{\tilde{A}t}) = \{0\}$ for $t \in [0, t_1]$.
Next, we show that there is a $t_0 \in (0, t_1)$ such that $\overline{\mathrm{Ran}(e^{\tilde{A}t_0})} = X$.
Since $(e^{At})_{t\in\mathbb{R}}$ is a C_0-group, $(e^{A^*t})_{t\in\mathbb{R}}$ is also a C_0-group. By applying the first part of the proof, we find that there exists a $t_2 > 0$ such that $\mathrm{Ker}(e^{\tilde{A}^*t}) = \{0\}$ for all $t \in (0, t_2]$. Thus for $t_0 = \min\{t_1, t_2\}$ we have

$$\mathrm{Ker}(e^{\tilde{A}t_0}) = \mathrm{Ker}(e^{\tilde{A}^*t_0}) = \{0\}. \tag{5.11}$$

The last equality is equivalent to the fact that the range of $e^{\tilde{A}t_0}$ is dense in X. Next, we show that this range is closed and thus $\mathrm{Ran}(e^{\tilde{A}t_0}) = X$.

Let $y_n \in \mathrm{Ran}(e^{\tilde{A}t_0})$ be a sequence converging to y. The sequence $x_n \in X$ defined by $e^{\tilde{A}t_0}x_m = y_m$ is a Cauchy sequence as well, since by equation (5.10),

$$\|x_n - x_m\| \leq \frac{1}{\delta}\|e^{\tilde{A}t_0}(x_n - x_m)\| = \frac{1}{\delta}\|y_n - y_m\|. \tag{5.12}$$

In the Hilbert space X, the Cauchy sequence x_n converges. Let x be the limit. Then

$$\begin{aligned}\|y - e^{\tilde{A}t_0}x\| &= \|y - y_n + e^{\tilde{A}t_0}x_n - e^{\tilde{A}t_0}x\| \\ &\leq \|y - y_n\| + \|e^{\tilde{A}t_0}\|\|x_n - x\| \to 0, \qquad \text{as } n \to \infty.\end{aligned}$$

So $y = e^{\tilde{A}t_0}x$ and thus $\mathrm{Ran}(e^{\tilde{A}t_0})$ is closed.
Summarizing we have that $\mathrm{Ker}(e^{\tilde{A}t_0}) = \{0\}$, and $\mathrm{Ran}(e^{\tilde{A}t_0}) = \overline{\mathrm{Ran}(e^{\tilde{A}t_0})} = X$. This means that $e^{\tilde{A}t_0}$ is invertible and thus $\tilde{A}$ generates a C_0-group on X. □

Now we focus on the set of states for which the semigroup is stable.

Definition 5.6 *Let A generate a semigroup on X, then we define the set of stable states $S(A)$ as follows,*

$$S(A) := \{x \in X \mid e^{At}x \to 0, \ \textit{if}\ t \to \infty\}.$$

By Theorem 4.8 we know that stability properties which hold for all elements of the space X, are shared by semigroups within the same Bergman class. In the following lemma, we state that also stability properties on a subset of X are shared within the Bergman classes.

Lemma 5.7 *Let $A \overset{B}{\sim} \tilde{A}$, and let the semigroup $(e^{\tilde{A}t})_{t\geq 0}$ be bounded, then*

$$S(A) = S(\tilde{A})$$

Proof: Since $A \overset{B}{\sim} \tilde{A}$, and $(e^{\tilde{A}t})_{t\geq 0}$ is bounded, the semigroup $(e^{At})_{t\geq 0}$ is bounded as well, see Theorem 4.8. Now, we show that $S(\tilde{A}) \subset S(A)$. Then by symmetry the equality holds. Let $x_0 \in S(\tilde{A})$. Since $\int_0^\infty \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds < \infty$, for every $\varepsilon > 0$, there exists a t_ε such that

$$\int_{t_\varepsilon}^\infty \frac{1}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds < \varepsilon. \tag{5.13}$$

Furthermore, the following inequality holds

$$\begin{aligned}\frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds \leq &\frac{1}{t}\int_0^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds \\ &+ \frac{1}{t}\int_0^t 2\|e^{\tilde{A}s}x_0\|^2 ds.\end{aligned}$$

With $x_0 \in S(\tilde{A})$ and using this inequality and Lemma 2.10, we have that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds &\leq \lim_{t\to\infty} \frac{1}{t}\int_0^{t_\varepsilon} 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 ds \\ &\quad + \lim_{t\to\infty} \frac{1}{t}\int_{t_\varepsilon}^t 2\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 + 0 \\ &\leq 0 + \lim_{t\to\infty} \int_{t_\varepsilon}^t \frac{2}{s}\|e^{As}x_0 - e^{\tilde{A}s}x_0\|^2 \\ &\leq 2\varepsilon,\end{aligned}$$

where we have used equation (5.13).
Since this holds for all $\varepsilon > 0$, we have shown that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|e^{As}x_0\|^2 ds = 0,$$

and so by Lemma 2.10 $\lim_{t\to\infty} e^{At}x_0 = 0$. Hence, $x_0 \in S(A)$. □

5.3 Characterization of Bergman equivalence classes on finite-dimensional state spaces

Although most of our results are formulated for infinite-dimensional state spaces X, for finite-dimensional spaces some additional results can be obtained. This is the topic of this section.
Let $M_n(\mathbb{C})$ be the space of all complex $n \times n$ matrices. A square matrix is called *Hurwitz* if all the eigenvalues have a strictly negative real part.
We begin by giving a characterization of the class $[-I]_B$ on the space $\mathbb{C}$.

Lemma 5.8 *On $\mathbb{C}^n$ we have that all Hurwitz matrices are in one equivalence class, i.e.,*

$$[-I]_B = \{A \in M_n(\mathbb{C}) \mid \text{Re } \lambda < 0, \textit{ for all } \lambda \in \sigma(A)\}.$$

Proof: If all eigenvalues of A have negative real part, then A generates an exponentially stable semigroup, just as $-I$. Since the operator $A + I$ is bounded, it follows from Lemma 4.18 that $A \overset{B}{\sim} -I$, and thus $A \in [-I]_B$.
If A has an eigenvalue λ with Re $\lambda \geq 0$, then for the corresponding eigenvector v the limit,

$$\lim_{t\to\infty} \int_0^t \|e^{\lambda t}v - e^{-t}v\|^2 \frac{1}{\tau} d\tau$$

does not exist. Thus this A is not an element of $[-I]_B$. This concludes the proof. □

So Hurwitz matrices form one class.
Now we examine matrices with unstable eigenvalues. For this we look at the Jordan decomposition $A = P^{-1}JP$, with J in Jordan form with increasing real part of the eigenvalues on the diagonal. We can write

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{5.14}$$

where the diagonal elements of J_1 have a negative real part, $\text{Re}(\lambda) < 0$, and the diagonal elements of J_2 have a non-negative real part, $\text{Re}(\lambda) \geq 0$. Corresponding to the decomposition (5.14), we split vector v as $v = \left[\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right]$.

Definition 5.9 *Let $A \in M_n(\mathbb{C})$ be written as $A = P^{-1}JP$, with J the Jordan canonical form, see (5.14). We define the set of unstable states $U(A)$ as follows,*

$$U(A) := \left\{ x \in \mathbb{C}^n \mid Px = \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right\}.$$

It is easy to see that

$$S(A) := \left\{ x \in \mathbb{C}^n \mid Px = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \right\}.$$

and $S(A) \oplus U(A) = \mathbb{C}^n$.

Lemma 5.10 *Let $n, m_i \in \mathbb{N}$ and $\alpha_i, \lambda_i \in \mathbb{C}$. Assume that $\alpha_i \neq 0$ and if $i \neq j$ then $(\lambda_i, m_i) \neq (\lambda_j, m_j)$, that is $\lambda_i \neq \lambda_j$ and/or $m_i \neq m_j$. Then the following two assertions are equivalent.*

(i)

$$\int_1^\infty \left| \sum_{i=0}^n \alpha_i t^{m_i} e^{\lambda_i t} \right|^2 \frac{1}{t} dt < \infty.$$

(ii)

$$\sum_{Re(\lambda_i) \geq 0} \alpha_i t^{m_i} e^{\lambda_i t} = 0, \qquad \textit{for all } t \geq 0.$$

Proof: $(i) \Rightarrow (ii)$: Assume that the integral is not equal to zero. Let μ be the biggest real part of λ_i. Then we can rewrite the sum as

$$\sum_{i=0}^n \alpha_i t^{m_i} e^{\lambda_i t} = e^{\mu t} \left(\tilde{\alpha} t^{\tilde{m}} + \sum_{j=0}^{\tilde{n}} \beta_j t^{m_j} e^{\nu_j t} \right),$$

where $\mathrm{Re}(\nu_j) \leq 0$ for all j. Since the integral is finite, we must have that $\mathrm{Re}(\mu) < 0$. Thus assertion (ii) holds.
$(ii) \Rightarrow (i)$: Since the sum over $\mathrm{Re}(\lambda_i) \geq 0$ is zero, for the integral we only have to look at the terms with $\mathrm{Re}(\lambda_i) < 0$. For all these terms individually the integral is finite, so the total integral is finite as well and assertion (i) holds. □

With this notation we can characterize $[A]_B$ for $A \in M_n(\mathbb{C})$. Two matrices are in the same equivalence class, if they have the same set of stable and unstable states and on the unstable states they have the same behaviour.

Theorem 5.11 *For A and $\tilde{A}$ matrices in $M_n(\mathbb{C})$, the following assertions are equivalent:*

(i) A and $\tilde{A}$ are in the same Bergman equivalence class, that is $A \overset{B}{\sim} \tilde{A}$.

(ii) $S(A) = S(\tilde{A})$, $U(A) = U(\tilde{A})$, and for all $x_u \in U(A)$ holds $Ax_u = \tilde{A}x_u$.

Proof: $(i) \Rightarrow (ii)$: For the orbits given by the semigroup, we can write

$$e^{At}x = \sum_i \alpha_i e^{\lambda_i t}\, t^{m_i}\, e_{k_i}, \tag{5.15}$$

with constants α, λ as eigenvalues of A and basis elements e_k.
From Lemma 5.10 we see that the Bergman distance is bounded, if all the terms with $\mathrm{Re}(\lambda) \geq 0$ cancel in the difference. That means $e^{At}x$ and $e^{\tilde{A}t}x$ both have these terms.
The matrix $e^{J_2 t}$ contains exactly the terms $e^{\lambda_i t}\, t^{m_i}$ with $\mathrm{Re}(\lambda_i) \geq 0$. So the matrix $e^{\tilde{J}_2 t}$ must contain the same terms elements. Since both matrices are ordered in the same way: $e^{J_2 t} = e^{\tilde{J}_2 t}$.
To show $S(A) = S(\tilde{A})$, we take $x_s \in S(A)$ and show $x_s \in S(\tilde{A})$.
For $x_s \in S(A)$, we can write $Px_s = \left(\begin{smallmatrix} v_1 \\ 0 \end{smallmatrix}\right)$ and

$$\begin{aligned}(e^{At} - e^{\tilde{A}t})x_s &= (P^{-1}e^{Jt}P - \tilde{P}^{-1}e^{\tilde{J}t}\tilde{P})x_s \\ &= P^{-1}\left(e^{Jt}(Px_s) - (P\tilde{P}^{-1})e^{\tilde{J}t}(\tilde{P}P^{-1})(Px_s)\right) \\ &= P^{-1}(e^{Jt}\begin{pmatrix} v_1 \\ 0 \end{pmatrix} - (P\tilde{P}^{-1})e^{\tilde{J}t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}),\end{aligned}$$

with $(\tilde{P}P^{-1})Px_s = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$.
No elements of $e^{\tilde{J}_2}$ may appear in the sum, so $w_2 = 0$. This means

$$\tilde{P}x_s = (\tilde{P}P^{-1})Px_s = \begin{pmatrix} w_1 \\ 0 \end{pmatrix}.$$

Thus $x_s \in S(\tilde{A})$ for all $x_s \in S(A)$. This means $S(A) \subset S(\tilde{A})$ and by symmetry $S(A) = S(\tilde{A})$.
To show $U(A) = U(\tilde{A})$, we take $x_u \in U(A)$ and show $x_u \in U(\tilde{A})$.
For $x_u \in U(A)$, we can write $Px_u = \left(\begin{smallmatrix} 0 \\ v_2 \end{smallmatrix}\right)$ and

$$\begin{aligned}
(e^{At} - e^{\tilde{A}t})x_u &= (P^{-1}e^{Jt}P - \tilde{P}^{-1}e^{\tilde{J}t}\tilde{P})x_u \\
&= P^{-1}\left(e^{Jt}(Px_u) - (P\tilde{P}^{-1})e^{\tilde{J}t}(\tilde{P}P^{-1})(Px_u)\right) \\
&= P^{-1}(e^{Jt}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} - (P\tilde{P}^{-1})e^{\tilde{J}t}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix})
\end{aligned}$$

with $(\tilde{P}P^{-1})Px_u = \left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$.
Both $e^{Jt}\left(\begin{smallmatrix} 0 \\ v_2 \end{smallmatrix}\right)$ and $P\tilde{P}^{-1}e^{\tilde{J}t}\left(\begin{smallmatrix} 0 \\ w_2 \end{smallmatrix}\right)$ only contain terms with nonnegative eigenvalues. All the terms must be cancelled. Thus,

$$e^{Jt}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} = P\tilde{P}^{-1}e^{\tilde{J}t}\begin{pmatrix} 0 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_2 \end{pmatrix}, \tag{5.16}$$

for some vector y_2.
In this equation, we can see that it is possible to write $P\tilde{P}^{-1}$ as a lower triangular block matrix,

$$P\tilde{P}^{-1} = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}.$$

This means $\tilde{P}P^{-1} = (P\tilde{P}^{-1})^{-1}$ is a lower triangular block matrix as well. So,

$$\tilde{P}x_u = \tilde{P}P^{-1}Px_u = \tilde{P}P^{-1}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ w_2 \end{pmatrix}. \tag{5.17}$$

Thus $x_u \in U(\tilde{A})$ and $U(A) \subset U(\tilde{A})$. By symmetry we have $U(A) = U(\tilde{A})$.
To conclude this part of the proof, we show that $Ax_u = \tilde{A}x_u$. For this we use equations (5.16) and (5.17).

$$\begin{aligned}
e^{At}x_u &= P^{-1}e^{Jt}Px_u = P^{-1}e^{Jt}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\
&= P^{-1}P\tilde{P}^{-1}e^{\tilde{J}t}\begin{pmatrix} 0 \\ w_2 \end{pmatrix} = \tilde{P}^{-1}e^{\tilde{J}t}\tilde{P}x_u = e^{\tilde{A}t}x_u.
\end{aligned}$$

Hence $Ax_u = \tilde{A}x_u$ for all $x_u \in U(A)$.

$(ii) \Rightarrow (i)$: There exist $x_s \in S(A)$ and $x_u \in U(A)$ such that $x = x_s + x_u$.

$$\begin{aligned}\int_0^\infty \|(e^{At} - e^{\tilde{A}t})x\|^2 \frac{1}{t} dt &= \int_0^\infty \|e^{At}x_s - e^{\tilde{A}t}x_s + e^{At}x_u - e^{\tilde{A}t}x_u\|^2 \frac{1}{t} dt \\ &\leq \int_0^\infty \|e^{At}x_s - e^{\tilde{A}t}x_s\|^2 \frac{1}{t} dt.\end{aligned}$$

State x_s is a stable state for both semigroups, so the integral from 1 to ∞ is bounded. The matrix difference $A - \tilde{A}$ is bounded, so the integral from 0 to 1 is bounded as well. Hence the Bergman distance is bounded. □

To illustrate Theorem 5.11 we give two examples. First we look at the semigroups on $\mathbb{C}$.

Example 5.12 *The C_0-semigroups on $\mathbb{C}$ are given by $(e^{at})_{t\geq 0}$ with $a \in \mathbb{C}$. The finite Bergman distance divides them into the following equivalence classes*

- *The class $[-1]_B$ contains the semigroups $(e^{at})_{t\geq 0}$ with $Re(a) < 0$.*
- *If $a \in \mathbb{C}$ $Re(a) \geq 0$, then the class $[a]_B$ only contains one semigroup.*

Next we look at the equivalence class of the C_0-semigroup generated by $A \in M_2(\mathbb{C})$.

Example 5.13 *The C_0-semigroup on $\mathbb{C}^2$ generated by*

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix},$$

is given by

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-3t} \end{pmatrix}, \qquad \text{for } t \geq 0.$$

The equivalence class $[A]_B$ contains the following semigroups

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\alpha t} \end{pmatrix}, \qquad \text{for } t \geq 0,$$

where $Re(\alpha) < 0$.

5.4 Conclusions

In this chapter we examined the Bergman distance, which was introduced in Chapter 4. We showed that the finite Bergman distance divides semigroups and cogenerators into equivalence classes. In Section 4.2, we had

already proved that within an equivalence class the semigroups share the same stability properties. Section 4.3 it is shown that the same holds for cogenerators. In Section 5.2 we extended these stability properties to element wise stability. Furthermore, we showed that a C_0-group shares its class with other C_0-groups.
In Section 5.3 a characterization of Bergman equivalence classes of finite-dimensional semigroups is given.
Looking at Definition 4.1, one can see that, in order to have a finite Bergman distance, it is required for two semigroups to have the same behaviour when $t \to \infty$. This notion is used to prove the stability results in Section 4.2, Section 4.3, and Section 5.2.
However, the measure $\frac{1}{t}$ puts an extra requirement on the two semigroups. In order to have finite Bergman distance, two semigroups have to behave in the same way when $t \to 0$. This notion gives rise to the following conjecture.

Conjecture 5.14 *Let A and $\tilde{A}$ generate a C_0-semigroup and let $A \overset{B}{\sim} \tilde{A}$, then*

$$D(A) = D(\tilde{A}). \tag{5.18}$$

Chapter 6

Norm relations using Laguerre polynomials

In Chapter 4 we defined the Bergman distance for semigroups, see Definition 4.3, and we defined the Bergman distance and the power sequences of cogenerators, see Definition 4.6. In Theorem 4.14 we showed that the Bergman distance of two semigroups is equal to the Bergman distance of the power sequences of their cogenerators.
In this chapter we examine the relation between the semigroup $(e^{At})_{t\geq 0}$ and the power sequence A_d^n using the methods of Section 4.4 and show other norm equalities. In Chapter 4 we used the Laguerre polynomials $L_n^{(1)}(t)$. In this chapter we use the more general Laguerre polynomials $L_n^{(\alpha)}(t)$. With these polynomials we can generalize Lemma 4.13 and obtain other norm equalities relating $(e^{At})_{t\geq 0}$ and A_d^n.
We begin with the definition of the generalized Laguerre polynomials, $L_n^{(\alpha)}(t)$.

Definition 6.1 *For $n \geq 0$, $t \geq 0$ and $\alpha > -1$, the Laguerre polynomials $L_n^{\alpha}(t)$ are given by, see [34, pag 100]:*

$$L_n^{\alpha}(t) = e^t \frac{t^{-\alpha}}{n!} \frac{d^n}{dt^n}(t^{n+\alpha}e^{-t}) = \sum_{k=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-t)^k}{k!(n-k)!}, \tag{6.1}$$

where Γ denotes the gamma function.

When we substitute $\alpha = 1$ the definition is the same as in equation (4.15).

6.1 Hilbert space and Parseval equality

In this section we follow a route similar to the one in Section 4.4. We begin with defining the inner product space H_α.

Definition 6.2 *For real $\alpha > -1$ let H_α denote the space of Lebesgue measurable functions f from $[0,\infty)$ to the Hilbert space X such that:*

$$\int_0^\infty \|f(t)\|_X^2 t^\alpha \, dt < \infty.$$

On H_α we define the following inner product:

$$\langle f, g\rangle_{H_\alpha} = \int_0^\infty \langle f(t), g(t)\rangle_X t^\alpha \, dt. \tag{6.2}$$

The following result is not hard to show.

Lemma 6.3 *The inner product space H_α defined in Definition 6.2 is a Hilbert space.*

To simplify further notation, we introduce three sequences.

Definition 6.4 *For $n \geq 0$ we define the sequences a_n, b_n and c_n as*

$$a_n = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}, \tag{6.3}$$

$$b_n = \frac{\Gamma(n+1)}{2^{\alpha+1}\Gamma(n+\alpha+1)}, \tag{6.4}$$

$$c_n = \sqrt{\frac{2^{\alpha+1}\Gamma(n+\alpha+1)}{\Gamma(n+1)}}. \tag{6.5}$$

Note that $2^{-(\alpha+1)/2}\, a_n = b_n\, c_n$.

Lemma 6.5 *Let $\alpha \in \mathbb{N}$. Then for the sequence c_n defined in equation (6.5), the following relation holds for $n \geq 0$:*

$$(n+1)^\alpha \leq \frac{c_n^2}{2^{\alpha+1}} \leq (n+\alpha)^\alpha. \tag{6.6}$$

Proof: From the definition of c_n, equation (6.5), we see that

$$\begin{aligned}\frac{c_n^2}{2^{\alpha+1}} &= \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \\ &= \prod_{k=n+1}^{n+\alpha} k,\end{aligned}$$

where we used the property of the gamma function that $\Gamma(n+1) = n\Gamma(n)$. Estimating the product terms k by the smallest and the largest term, gives relation (6.6). □

The following lemma is a generalization of Lemma 4.12.

Lemma 6.6 *Let H_α be the Hilbert space defined by Definition 6.2 and then let $\{e_m\}_{m\in\mathbb{N}}$ be an orthonormal basis of X. The vectors $\varphi_{n,m}$ defined by:*

$$\varphi_{n,m}(t) = a_n e^{-t/2} L_n^\alpha(t) e_m, \qquad n, m \geq 0, \tag{6.7}$$

form an orthonormal basis in H_α.

Proof: We begin by showing that the sequence $\{\varphi_{n,m}\}_{n,m=0}^{\infty}$ is orthonormal in H_α. Using equation (6.2), we find:

$$\begin{aligned}
\langle \varphi_{n,m}, \varphi_{\nu,\mu} \rangle_{H_\alpha} &= \int_0^\infty \langle a_n e^{-t/2} L_n^\alpha(t) e_m, a_\nu e^{-t/2} L_\nu^\alpha(t) e_\mu \rangle_X t^\alpha \, dt \\
&= a_n a_\nu \int_0^\infty e^{-t} t^\alpha L_n^\alpha(t) L_\nu^\alpha(t) dt \, \langle e_m, e_\mu \rangle_X \\
&= a_n a_\nu \, \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \, \delta_{n\nu} \delta_{m\mu} = \delta_{n\nu} \delta_{m\mu},
\end{aligned}$$

where we use the orthogonality of the Laguerre polynomials in $L^2(0,\infty)$ with weight $\tau e^{-\tau}$, see [34, pag 99].
Next we show that the sequence $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is maximal in H_α. If $h \in H_\alpha$ is orthogonal to every $\varphi_{n,m}$, then for all n and $m \geq 0$:

$$\langle \varphi_{n,m}, h \rangle_{H_\alpha} = \int_0^\infty \langle a_n e^{-t/2} L_n^\alpha(t) e_m, h(t) \rangle_X t^\alpha \, dt = 0.$$

Using the maximality of $\{e^{-t/2} t^\alpha L_n^\alpha(t)\}_{n\geq 0}$ in $L^2(0,\infty)$, we conclude that for all $m \geq 1$,

$$\langle e_m, h(t) \rangle_X = 0 \quad \text{almost everywhere.}$$

This, combined with the maximality of $\{e_m\}_{m\in\mathbb{N}}$ in X, leads to the conclusion that the function $h(t) = 0$ almost everywhere. So $h = 0$ in H_α and $\{\varphi_{n,m}\}_{n,m=1}^{\infty}$ is maximal. □

In particular, Lemma 6.6 gives us the following Parseval equality.

$$\|f\|_{H_\alpha}^2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |\langle f, \varphi_{n,m} \rangle_{H_\alpha}|^2. \tag{6.8}$$

6.2 Generalization of Lemma 4.13

Using the Laguerre polynomials, we can transform the semigroup into the cogenerator. Doing this, we obtain a generalization of Lemma 4.13. Note that only in this section we restrict α to be a nonnegative integer.

Lemma 6.7 *Let n and $\alpha \in \mathbb{N}$ and let A generate a semigroup with growth bound $\omega < 1$. Further let $L_n^\alpha(t)$ be the Laguerre polynomials, see (6.1). Then for every $x_0 \in D(A^{\alpha-1})$ the following relation holds,*

$$
\begin{aligned}
-\int_0^\infty 2^{\alpha-1} e^{-t/2} L_n^\alpha(t) e^{At/2} x_0 dt &= \\
& A_d^{n+\alpha}\left(A-I\right)^{\alpha-1} x_0 - \sum_{\ell=0}^{\alpha-1} \frac{(n+\alpha)!}{\ell!(n+\alpha-\ell)!} 2^\ell \left(A-I\right)^{\alpha-1-\ell} x_0. \qquad (6.9)
\end{aligned}
$$

Proof: Using equation (6.1) we can rewrite the left-hand side of equation (6.9) as follows,

$$
\begin{aligned}
-\int_0^\infty & 2^{\alpha-1} e^{-t/2} L_n^\alpha(t) e^{At/2} x_0 dt \\
&= -2^{\alpha-1} \int_0^\infty e^{-t/2} \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-t)^k}{k!(n-k)!} e^{At/2} x_0 dt \\
&= 2^{\alpha-1} \sum_{k=0}^n (-1)^{k+1} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)(n-k)!} \int_0^\infty \frac{t^k}{k!} e^{-t/2} e^{At/2} x_0 dt \\
&= 2^{\alpha-1} \sum_{k=0}^n (-1)^{k+1} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)(n-k)!} \left(\frac{1}{2}I - \frac{1}{2}A\right)^{-(k+1)} x_0 \\
&= \sum_{k=0}^n 2^{k+\alpha} \frac{(n+\alpha)!}{(k+\alpha)!(n-k)!} \left(A-I\right)^{-(k+1)} x_0.
\end{aligned}
$$

In the second last step we used

$$
\int_0^\infty \frac{t^k}{k!} e^{-t/2} e^{At/2} dt = \left(\frac{1}{2}I - \frac{1}{2}A\right)^{-(k+1)}, \qquad (6.10)
$$

see [22, Proposition 3.3.5, pag 76]. Now we introduce $\ell = k + \alpha$.

$$
\begin{aligned}
-\int_0^\infty & 2^{\alpha-1} e^{-t/2} L_n^\alpha(t) e^{At/2} x_0 dt \\
&= \sum_{\ell=\alpha}^{n+\alpha} 2^\ell \frac{(n+\alpha)!}{\ell!(n+\alpha-\ell)!} (A-I)^{\alpha-1-\ell} x_0 \\
&= (A-I)^{\alpha-1} \left[\sum_{\ell=0}^{n+\alpha} \frac{(n+\alpha)!}{\ell!(n+\alpha-\ell)!} 2^\ell (A-I)^{-\ell} \right. \\
&\qquad \left. - \sum_{\ell=0}^{\alpha-1} \frac{(n+\alpha)!}{\ell!(n+\alpha-\ell)!} 2^\ell (A-I)^{-\ell} \right] x_0 \\
&= (A-I)^{\alpha-1} \left[(I + 2(A-I)^{-1})^{n+\alpha} \right. \\
&\qquad \left. - \sum_{\ell=0}^{\alpha-1} \frac{(n+\alpha)!}{\ell!(n+\alpha-\ell)!} 2^\ell (A-I)^{-\ell} \right] x_0 \\
&= A_d^{n+\alpha} (A-I)^{\alpha-1} x_0 - \sum_{\ell=0}^{\alpha-1} \frac{(n+\alpha)!}{\ell!(n+\alpha-\ell)!} 2^\ell (A-I)^{\alpha-1-\ell} x_0.
\end{aligned}
$$

where we used the binomial theorem in the second last step and equation (1.17) in the last step. Thus equation 6.14 holds. □

Remark 6.8 *To give some insight into equation (6.9), we specify the equation for some values of α. First the case $\alpha = 0$:*

$$
-\int_0^\infty \frac{1}{2} e^{-t/2} L_n^0(t) e^{At/2} x_0 dt = A_d^n (A-I)^{-1} x_0. \tag{6.11}
$$

The case $\alpha = 1$:

$$
-\int_0^\infty e^{-t/2} L_n^1(t) e^{At/2} x_0 dt = A_d^{n+1} x_0 - x_0. \tag{6.12}
$$

This is the same equation as equation (4.19). Hence Lemma 4.13 is a special case of Lemma 6.7.
The case $\alpha = 2$:

$$
\begin{aligned}
-\int_0^\infty 2e^{-t/2} L_n^2(t) e^{At/2} x_0 dt =& A_d^{n+2} (A-I) x_0 - (A-I) x_0 \\
&+ 2(n+2) x_0.
\end{aligned} \tag{6.13}
$$

Note that for every step in α, on the right-hand side an extra term is added.

In Chapter 4 we derived equation (6.12) in Lemma 4.13. In Theorem 4.14 we used this equation to obtain a norm relation between semigroups and their cogenerators. To deal with the $-x_0$ term on the right-hand side of equation (6.12), we compared the difference of two semigroups.

If such a norm relation can be obtain using equation (6.13) is not a easy question, since there is no easy trick to deal with the two extra terms on the right-hand side. The same holds for $\alpha \geq 3$.

On the other hand, with equation (6.11) it is possible to obtain a norm equality between the semigroup and the cogenerator. In Section 6.3 this norm equality is derived as a special case of Theorem 6.11.

6.3 Norm equality

In Section 6.2 we derived a relation between the semigroup and the cogenerator in Lemma 6.7. In the case $\alpha = 1$ we used equation 6.9 to obtain a norm relation, see Theorem 4.14.

In this section we derive a relation between the semigroup and the cogenerator as well. However, now we add a t^α term to the integral on the left-hand side of equation (6.14). In this way we can obtain a norm relation for general α. Note in this section $\alpha \in \mathbb{R}$.

As in Section 6.2, we use the Laguerre polynomials to transform the semigroup into the cogenerator.

Lemma 6.9 *Let $n \geq 0$, let $\alpha > -1$ and let A generate a semigroup with growth bound $\omega < 1$. Further let $L_n^\alpha(t)$ be the Laguerre polynomials, see (6.1), and let $\{b_n\}_{n\geq 0}$ be given by (6.4). Then for every $x_0 \in X$ the following relation holds,*

$$\int_0^\infty b_n e^{-t/2} t^\alpha L_n^\alpha(t) e^{At/2} x_0 dt = A_d^n (I - A)^{-(\alpha+1)} x_0, \qquad x_0 \in X. \quad (6.14)$$

Proof: By substituting equation (6.4) and (6.1) in the above we find that

$$\begin{aligned}
&\int_0^\infty b_n e^{-t/2} t^\alpha L_n^\alpha(t) e^{At/2} x_0 dt \\
&= \int_0^\infty \frac{\Gamma(n+1)}{2^{\alpha+1}\Gamma(n+\alpha+1)} e^{-t/2} t^\alpha \sum_{k=0}^n \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} \frac{(-t)^k}{k!(n-k)!} e^{At/2} x_0 dt \\
&= \sum_{k=0}^n \frac{(-1)^k}{2^{\alpha+1}} \frac{n!}{k!(n-k)!} \int_0^\infty \frac{t^{k+\alpha}}{\Gamma(k+\alpha+1)} e^{-t/2} e^{At/2} x_0 dt \\
&= \sum_{k=0}^n \frac{(-1)^k}{2^{\alpha+1}} \frac{n!}{k!(n-k)!} \left(\frac{1}{2}I - \frac{1}{2}A\right)^{-(k+\alpha+1)} x_0,
\end{aligned}$$

where we used [22, Proposition 3.3.5]. Thus we have

$$\begin{aligned}
\int_0^\infty b_n e^{-t/2} t^\alpha L_n^\alpha(t) e^{At/2} x_0 dt &= \sum_{k=0}^n 2^k \frac{n!}{k!(n-k)!} (A-I)^{-k} (I-A)^{-(\alpha+1)} x_0 \\
&= \left(I + 2(A-I)^{-1}\right)^n (I-A)^{-(\alpha+1)} x_0 \\
&= A_d^n (I-A)^{-(\alpha+1)} x_0,
\end{aligned}$$

where we used (1.17). Concluding, we see that equation (6.14) holds. □

Remark 6.10 *If we take $\alpha = 0$, equation (6.14) is the same as equation (6.11).*

Combining the equality (6.14) with the Parseval identity, we obtain a element-wise norm equality between the semigroup and the cogenerator.

Theorem 6.11 *Let A be the infinitesimal generator of the C_0-semigroup $(e^{At})_{t\geq 0}$ with growth bound $\omega < 1$ and let A_d be its cogenerator. Then for all $\alpha > -1$ and all $x_0 \in X$, there holds*

$$\int_0^\infty \|e^{At} x_0\|_X^2 t^\alpha dt = \sum_{n=0}^\infty c_n^2 \left\| A_d^n (I-A)^{-(\alpha+1)} x_0 \right\|_X^2 . \tag{6.15}$$

Here c_n is the constant defined in Definition 6.4.

Proof: First, we substitute $\tau = 2t$ and we write the left-hand side of equation (6.15) as a norm in H_α, see Definition 6.2. Thus the left-hand side of equation (6.15) becomes

$$\begin{aligned}\int_0^\infty \|e^{At}x_0\|_X^2 t^\alpha dt &= 2^{-(\alpha+1)}\int_0^\infty \|e^{A\tau/2}x_0\|_X^2\tau^\alpha d\tau\\ &= 2^{-(\alpha+1)}\left\|e^{A\tau/2}x_0\right\|_{H_\alpha}^2\\ &= \sum_{n=0}^\infty\sum_{m=0}^\infty |\langle 2^{-(\alpha+1)/2}e^{A\tau/2}x_0, \varphi_{n,m}(\tau)\rangle_{H_\alpha}|^2,\end{aligned}$$

where we used the Parseval identity on H_α, see equation (6.8).
Zooming in on the inner product, and applying equation (6.7) we find

$$\begin{aligned}&\left\langle 2^{-(\alpha+1)/2}e^{A\tau/2}x_0,\varphi_{n,m}(\tau)\right\rangle_H\\ &\quad= \int_0^\infty \left\langle 2^{-(\alpha+1)/2}e^{A\tau/2}x_0, a_n e^{-\tau/2}L_n^\alpha(\tau)e_m\right\rangle_X \tau^\alpha d\tau\\ &\quad= c_n\int_0^\infty \left\langle b_n e^{-\tau/2}\tau^\alpha L_n^\alpha(\tau)e^{A\tau/2}x_0, e_m\right\rangle_X d\tau\\ &\quad= c_n\left\langle \int_0^\infty b_n e^{-\tau/2}\tau^\alpha L_n^\alpha(\tau)e^{A\tau/2}x_0 d\tau, e_m\right\rangle_X\\ &\quad= c_n\left\langle A_d^n(I-A)^{-(\alpha+1)}x_0, e_m\right\rangle_X,\end{aligned}$$

where we applied Lemma 6.9 in the last step.
We zoom out again and use the Parseval equation of X for the orthonormal basis $\{e_m\}_{m\in\mathbb{N}}$.

$$\begin{aligned}\int_0^\infty \|e^{At}x_0\|_X^2 t^\alpha dt &= \sum_{n=0}^\infty\sum_{m=0}^\infty \left|c_n\left\langle A_d^n(I-A)^{-(\alpha+1)}x_0, e_m\right\rangle_X\right|^2\\ &= \sum_{n=0}^\infty c_n^2\sum_{m=0}^\infty \left|\left\langle A_d^n(I-A)^{-(\alpha+1)}x_0, e_m\right\rangle_X\right|^2\\ &= \sum_{n=0}^\infty c_n^2\left\|A_d^n(I-A)^{-(\alpha+1)}x_0\right\|_X^2\end{aligned}$$

Thus equation (6.15) holds. □

Remark 6.12 *In equation (6.15) there are two weights, the continuous-time weight t^α and the discrete-time weight $c_n^2 \approx 2^{\alpha+1}n^\alpha$, see Lemma 6.5. Note that they depend in the same way on α.*

6.4 Conclusions

In this chapter we generalized the method introduced in Section 4.4. For this we used the Laguerre polynomials $L_n^\alpha(t)$ with $\alpha > -1$. We defined the inner product space H_α and derived the Parseval equality on H_α.
These tools were used to obtain a relation between the semigroup and the cogenerator in Lemma 6.7, which is a generalization of Lemma 4.13
In Lemma 6.9 we obtained a different relation between the semigroup and the cogenerator by introducing a weight t^α. We used this relation to relate weighted norms on the semigroup to weighted norms on the cogenerator, see Theorem 6.11.

Chapter 7

Growth relation cogenerator and inverse generator

In the previous chapters we have assumed that X is a Hilbert space. In Section 1.3 we have given some results for Banach spaces. Among others we showed that the worst growth for powers of the cogenerator is $\sqrt{n}$. First we looked at the relation between $(e^{At})_{t\geq 0}$ and $(A_d^n)_{n\in\mathbb{N}}$ (semigroup and powers of cogenerator), but now we include as well the growth of the semigroup generated by the inverse of A, i.e. by A^{-1}.
An important question regarding $(e^{A^{-1}t})_{t\geq 0}$ was stated by deLaubenfels in [13]: Does A^{-1} generates a bounded C_0-semigroup, if A generates a bounded C_0-semigroup? In [40], Zwart gave a Banach space counterexample. However, for Hilbert spaces this question is still open.
In this chapter we show that the growth of the cogenerator is directly related to the semigroup generated by A^{-1}. Thus it is possible to construct a Banach space and an infinitesimal generator of a bounded semigroup on this Banach space such that

$$\|e^{A^{-1}t}\| \approx \sqrt{t}. \tag{7.1}$$

For Hilbert spaces we show that if A and A^{-1} generate a bounded semigroup then the powers of the cogenerator are bounded. This result was first proved in [2]. We present a new proof with a sharper estimate in the sup-norm of $\|A_d^n\|$.

7.1 Exponentially stable semigroups on a Banach space

The following theorem shows that for an infinitesimal generator A of an exponentially stable C_0-semigroup growth bounds on the power sequence of the cogenerator $(A_d^n)_{n\in\mathbb{N}}$, indicate similar growth bounds for the semigroup generated by the inverse $\left(e^{A^{-1}t}\right)_{t\geq 0}$.

Theorem 7.1 *Assume that for every A that generates an exponentially stable C_0-semigroup the following estimate holds*

$$\|A_d^n\| \leq M_A g(n), \qquad n \in \mathbb{N}, \tag{7.2}$$

where g is a monotonically non-decreasing function, not depending on A, i.e, $0 < g(\alpha) \leq g(\beta)$ for all $0 \leq \alpha \leq \beta$, and M_A a constant not depending on n. If A^{-1} generates a C_0-semigroup, then for the C_0-semigroup $\left(e^{A^{-1}t}\right)_{t\geq 0}$ a similar estimate holds, i.e,

$$\|e^{A^{-1}t}\| \leq \tilde{M}_A g(\frac{2et}{\omega}), \qquad t \geq 0. \tag{7.3}$$

Here the (positive) constant ω is such that $-\omega$ is larger than the growth bound of A, that is there exists an $M > 0$ such that for all $t \geq 0$ there holds $\|e^{At}\| \leq Me^{-\omega t}$.

Proof: Recall from Lemma 1.1 in Section 1.3 that in Banach spaces the growth of the powers of the cogenerator is bounded by $\sqrt{n}$. Thus without loss of generality we may assume that $g(n) \leq c_0(1 + \sqrt{n})$.
For the ω as introduced at the end of the theorem, it is easy to see that A_0 defined by

$$A_0 = 2\omega^{-1}A + I.$$

generates an exponentially stable C_0-semigroup. Furthermore, its Cayley transform satisfies $(A_0)_d = (A_0 + I)(A_0 - I)^{-1} = I + \omega A^{-1}$. Hence

$$\|e^{\omega A^{-1}t}\| = e^{-t}\|e^{(A_0)_d t}\|, \qquad t \geq 0. \tag{7.4}$$

Since A_0 generates an exponentially stable C_0-semigroup, the powers of $(A_0)_d$ satisfy the estimate (7.2). From this and using Stirlings estimate,

$$n! \geq \left(\frac{n}{e}\right)^n \sqrt{2\pi n}, \qquad \text{for all } n \in \mathbb{N},$$

we have for $t \geq 1$

$$\begin{aligned}
\|e^{(A_0)_d t}\| &\leq \sum_{n=0}^{\infty} \frac{t^n \|(A_0)_d^n\|}{n!} \leq M_{A_0} \sum_{n=0}^{\infty} \frac{g(n)t^n}{n!} \\
&\leq c_1 \left[\sum_{n \geq 2et} \left(\frac{et}{n}\right)^n \frac{g(n)}{\sqrt{n}} + \sum_{n \leq 2et} \frac{g(n)t^n}{n!} \right] \\
&\leq c_1 \left[\sum_{n=1}^{\infty} \frac{g(n)}{2^n \sqrt{n}} + g(2et) \sum_{n=0}^{\infty} \frac{t^n}{n!} \right] = c_2 + c_1 g(2et) e^t. \qquad (7.5)
\end{aligned}$$

Here we have used that $g(n) \leq c_0(1+\sqrt{n})$. From equation (7.5) and (7.4) we obtain estimate equation (7.3). □

There are several consequences of this result.
We start with the relation between the boundedness of A_d and the boundedness of $\left(e^{A^{-1}t}\right)_{t\geq 0}$.

Corollary 7.2 *Suppose that on the Banach space X there exists the generator A of an exponentially stable semigroup, such that its inverse operator A^{-1} does not generate a bounded C_0-semigroup. Then there exists an operator A_0 that generates an exponentially stable C_0-semigroup, such that the power sequence of its cogenerator $((A_0)_d^n)_{n\in\mathbb{N}}$ is not bounded.*

Proof: If for every generator A_0 of a exponentially stable C_0-semigroup its cogenerator $(A_0)_d$ is bounded, then we can choose $g(n) \equiv 1$, see (7.2). Thus by Theorem 7.1, A^{-1} generates a bounded C_0-semigroup. This provides a contradiction with the assumptions. □

We can combine Theorem 7.1 with results from Chapter 1 and 3. Doing this, we obtain different proofs for estimates found in the literature. The first result can be found in [40], but it also follows from Theorem 7.1 and Lemma 1.3.

Corollary 7.3 *Let X be a Banach space and let A generate an exponentially stable C_0-semigroup on X, then the following growth estimate holds*

$$\|e^{A^{-1}t}\| \leq 1 + M_0 t^{\frac{1}{4}}, \qquad t \geq 0,$$

with M_0 independent of t.

Another result can be found in [39]. For Hilbert spaces it was shown by Gomilko that g in (7.2) can be chosen as $\ln(n+1)$. This is an extension of Theorem 3.1 in Chapter 3. Combining this with Theorem 7.1 gives the estimate found in [39].

Corollary 7.4 *Let X be a Hilbert space and let A generate an exponentially stable C_0-semigroup on X, then the following growth estimate holds.*

$$\|e^{A^{-1}t}\| \leq M_0 \ln(t+2), \qquad t \geq 0,$$

with M_0 independent of t.

The following theorem can be seen as a partial reverse implication of Theorem 7.1.

Theorem 7.5 *Assume that for every A that generates an exponentially stable C_0-semigroup the following estimate holds*

$$\|e^{A^{-1}t}\| \leq \tilde{M}_A g(t),$$

where g is a monotonically non-decreasing function, not depending on A, i.e, $0 < g(\alpha) \leq g(\beta)$ for all $0 \leq \alpha \leq \beta$, and M_A a constant not depending on t. Then for every A which is the generator of an exponentially stable semigroup satisfying $\|e^{At}\| \leq Me^{-\omega t}$ for $\omega > 1$, there exists for all $\alpha > 1$ an $M_{\alpha,A}$ such that

$$\|A_d^n\| \leq M_{\alpha,A} g(\alpha n).$$

Proof: By Lemma 1.3 we see that without loss of generality we may assume that $g(t) \leq 1 + M_0 t^{\frac{1}{4}}$.
Let $\alpha > 1$ be given. Since for $\alpha > 1$, we have that $\alpha - 1 - \log(\alpha) > 0$, there exists an $\varepsilon \in (0,1)$ such that

$$\alpha\varepsilon < \alpha - 1 - \log(\alpha). \tag{7.6}$$

Secondly, the function $e^{-(1-\varepsilon)t}t^{n-1}, t > 0$ has a maximum at $\tau = \frac{n-1}{1-\varepsilon}$ and is decreasing for $t > \tau$. We choose now

$$t_1 = \alpha(1-\varepsilon)\tau = \alpha(n-1). \tag{7.7}$$

Since $\alpha > 1$, and since equation (7.6) holds, we have that $t_1 > \tau$.
We define $A_1 = \frac{1}{2}(A + I)$. By the assumption on A, we have that A_1 generates an exponentially stable C_0-semigroup. Furthermore, we have that

$$A_d = (A+I)(A-I)^{-1} = 2A_1(2A_1 - 2I)^{-1} = (I - A_1^{-1})^{-1}.$$

The powers of A_d satisfy the norm estimates

$$\begin{aligned}
\|A_d^n\| &= \|(I - A_1^{-1})^{-n}\| \\
&\leq \frac{1}{(n-1)!}\int_0^\infty e^{-t}\|e^{A_1^{-1}t}\|t^{n-1}dt \\
&\leq \frac{1}{(n-1)!}\int_0^\infty e^{-t}\tilde{M}_{A_1}g(t)t^{n-1}dt \\
&= \frac{\tilde{M}_{A_1}}{(n-1)!}\int_0^\infty e^{-t}g(t)t^{n-1}dt \\
&= \frac{\tilde{M}_{A_1}}{(n-1)!}\int_0^{t_1} e^{-t}g(t)t^{n-1}dt + \frac{\tilde{M}_{A_1}}{(n-1)!}\int_{t_1}^\infty e^{-t}g(t)t^{n-1}dt. \qquad (7.8)
\end{aligned}$$

The estimate for the time interval from 0 to t_1 is simple using the fact that g is non-decreasing

$$\frac{\tilde{M}_{A_1}}{(n-1)!}\int_0^{t_1} e^{-t}g(t)t^{n-1}dt \leq \frac{\tilde{M}_{A_1}g(t_1)}{(n-1)!}\int_0^\infty e^{-t}t^{n-1}dt = \tilde{M}_{A_1}g(t_1). \qquad (7.9)$$

For the estimate for the time interval from t_1 to ∞ we use two observations. First Stirling's approximation for $(n-1)!$

$$(n-1)! \geq \sqrt{2\pi(n-1)}\left(\frac{n-1}{e}\right)^{n-1}.$$

Secondly, by the choice of t_1, see (7.7), we have that

$$e^{-(1-\varepsilon)t}t^{n-1} \leq e^{-(1-\varepsilon)\alpha(n-1)}(\alpha(n-1))^{n-1}, \qquad \text{for } t \geq t_1.$$

Combining, we find

$$\begin{aligned}
&\frac{\tilde{M}_{A_1}}{(n-1)!}\int_{t_1}^{\infty} e^{-t}g(t)t^{n-1}dt \\
&\quad\leq \frac{\tilde{M}_{A_1}}{\sqrt{2\pi(n-1)}}\left(\frac{e}{n-1}\right)^{n-1}\int_{t_1}^{\infty} e^{-t}g(t)t^{n-1}dt \\
&\quad\leq \frac{\tilde{M}_{A_1}}{\sqrt{2\pi(n-1)}}\left(\frac{e}{n-1}\right)^{n-1}\int_{t_1}^{\infty} e^{-(1-\varepsilon)t}t^{(n-1)}e^{-\varepsilon t}g(t)dt \\
&\quad\leq \frac{\tilde{M}_{A_1}}{\sqrt{2\pi(n-1)}}\left(\frac{e}{n-1}\right)^{n-1} e^{-(1-\varepsilon)\alpha(n-1)}(\alpha(n-1))^{n-1} \\
&\qquad\qquad \int_{t_1}^{\infty} e^{-\varepsilon t}g(t)dt \\
&\quad= \frac{\tilde{M}_{A_1}}{\sqrt{2\pi(n-1)}}\left(e^{(-(1-\varepsilon)\alpha+1)}\alpha\right)^{n-1}\int_{t_1}^{\infty} e^{-\varepsilon t}g(t)dt.
\end{aligned}$$

Using equation (7.6) we have that $e^{(-(1-\varepsilon)\alpha+1)}\alpha < 1$. Thus we have that

$$\frac{\tilde{M}_{A_1}}{(n-1)!}\int_{t_1}^{\infty} e^{-t}g(t)t^{n-1}dt \leq \frac{\tilde{M}_{A_1}}{\sqrt{2\pi(n-1)}}\int_{t_1}^{\infty} e^{-\varepsilon t}g(t)dt = M_\alpha \tag{7.10}$$

with M_α independent of n. Combining the estimates (7.8), (7.9) and (7.10) gives

$$\|A_d^n\| \leq \tilde{M}_{A_1}g(t_1) + M_\alpha.$$

Since $t_1 = \alpha(n-1)$ and since g is non-decreasing, we can find a constant $M_{\alpha,A}$ such that $\|A_d^n\| \leq \tilde{M}_{\alpha,A}g(\alpha n)$ which proves the result. □

7.2 Bounded semigroups on a Hilbert space

In the previous section, we have investigated the growth of $(A_d^n)_{n\in\mathbb{N}}$ and $\left(e^{A^{-1}t}\right)_{t\geq 0}$ under the condition that A generates an exponentially stable C_0-semigroup on Banach space X.

In this section we take X to be a Hilbert space. Furthermore, we assume that A and A^{-1} generate a bounded semigroup.

In this section we use the notation $R(A,s) = (sI - A)^{-1}$.

For a bounded operator A that generates a bounded C_0-semigroup on a Hilbert space, we know that the power sequence of its cogenerator is bounded as well, see Lemma 1.6. However, if A generates an bounded semigroup, but

the generator itself A is unbounded, then it is unknown whether the above result holds. However, if A and A^{-1} both generate a bounded semigroup, then the power sequence of the cogenerator is bounded.
We present a new proof of this theorem, for that we need some facts on Lyapunov equations.

Theorem 7.6 *Let A generate a bounded C_0-semigroup and let A_d denote its cogenerator. Suppose that the inverse of A, A^{-1}, exists and generates a bounded C_0-semigroup.*

1. *For $x \in X$ and $\lambda > 1$ define the functional $G_A(x;\lambda)$ as*

$$G_A(x;\lambda) := \frac{1}{\lambda^2} \sum_{n=0}^{\infty} \left[\|R^n(A,\lambda)x\|^2 + \|R^n(A^{-1},\lambda)x\|^2\right] \left(\lambda^2 - 1\right)^n . \tag{7.11}$$

Then

$$G_A(x;\lambda) + \frac{(\lambda-1)}{\lambda^2}\|x\|^2 = \frac{2(1-r^2)}{(1+r^2)} \sum_{n=0}^{\infty} \|A_d^n x\|^2 r^{2n}, \quad x \in X, \tag{7.12}$$

where $\lambda = \frac{1+r^2}{1-r^2}$ and $r \in (0,1)$.

2. *The power sequence $(A_d^n)_{n\in\mathbb{N}}$ is bounded and*

$$\|A_d^n\| \le \frac{e}{2}\left(\frac{1}{4} + M^2 + M_1^2\right), \tag{7.13}$$

where $M = \sup_{t>0} \|e^{At}\|$ and $M_1 = \sup_{t>0} \|e^{A^{-1}t}\|$.

Proof: The proof of the first item can found in [20, Theorem 2]. However, we present a new proof using Lyapunov equations.

Proof of part 1. Since A generates a bounded semigroup we have by the Hille-Yosida Theorem that there exists a constant $M \ge 1$, such that

$$\|R^n(A,\lambda)\| \le M\lambda^{-n}, \qquad \lambda > 0, \quad n \in \mathbb{N}.$$

This means that for $\lambda > 1$ we can take the following sum:

$$\sum_{n=0}^{\infty} \left\| \left(\sqrt{\lambda^2 - 1}\right)^n R^n(A,\lambda) \right\|^2 \le \sum_{n=0}^{\infty} M^2 \left(\frac{\lambda^2-1}{\lambda^2}\right)^n = M^2\lambda^2. \tag{7.14}$$

Using Lemma 2.14, Theorem 2.15 and Remark 2.17 we know that there exists an unique solution for Lyapunov equation (2.45).

Furthermore, this solution satisfies

$$\langle x, P_1 x\rangle = \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R(A,\lambda)^n x\|^2. \tag{7.15}$$

Similar equations hold for A^{-1}. Thus, there exists an operator $P_2 \in \mathcal{L}(X)$, being the unique solution of equation (2.46), satisfying

$$\langle x, P_2 x\rangle = \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R(A^{-1},\lambda)^n x\|^2. \tag{7.16}$$

By Lemma 2.27, the operator

$$P_V := \frac{1}{2\lambda}(P_1 + P_2 + \lambda I - I),$$

is the unique solution of equation (2.47). Moreover, since P_1, P_2 are positive, and since $\lambda > 1$ we find that $P_V > 0$, and so by Remark 2.18 there holds

$$\langle x, P_V x\rangle = \sum_{n=0}^{\infty} \left(\frac{\lambda - 1}{\lambda + 1}\right)^n \|A_d^n x\|^2. \tag{7.17}$$

Writing equation (2.49) as an inner product

$$\langle x, \frac{1}{2\lambda}(P_1 + P_2 + \lambda I - I)x\rangle = \langle x, P_V x\rangle.$$

and substituting equation (7.15), (7.16), and (7.17), we find that

$$\sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R(A,\lambda)^n x\|^2 + \sum_{n=0}^{\infty} (\lambda^2 - 1)^n \|R(A^{-1},\lambda)^n x\|^2 \\ + (\lambda - 1)\|x\|^2 = 2\lambda \sum_{n=0}^{\infty} \left(\frac{\lambda - 1}{\lambda + 1}\right)^n \|A_d^n x\|^2.$$

Dividing by λ^2 and substituting $\lambda = (1 + r^2)/(1 - r^2)$ on the right-hand side, we have proved equation (7.12).

Proof of part 2. Since A and A^{-1} generate a bounded C_0-semigroup, we have by the Hille-Yosida Theorem that there exist constants $M \geq 1$, $M_1 \geq 1$, such that

$$\|R^n(A,\lambda)\| \leq M\lambda^{-n}, \quad \|R^n(A^{-1},\lambda)\| \leq M_1\lambda^{-n}, \qquad \lambda > 0, \quad n \in \mathbb{N}. \tag{7.18}$$

Substituting this in (7.11), we find that

$$\begin{aligned} G_A(x,\lambda) &\le \frac{M^2+M_1^2}{\lambda^2}\sum_{n=0}^{\infty}\left[\frac{\lambda^2-1}{\lambda^2}\right]^n \|x\|^2 \\ &= \frac{M^2+M_1^2}{\lambda^2}\lambda^2\|x\|^2 = (M^2+M_1^2)\|x\|^2. \end{aligned} \tag{7.19}$$

Next using elementary calculus, we find that for any $x \in X$

$$r \in (0,1) \Rightarrow \lambda = \frac{1+r^2}{1-r^2} \in (1,\infty),$$

$$\frac{(\lambda-1)}{\lambda^2} \le \frac{1}{4}, \quad \lambda > 1$$

and

$$\frac{2(1+r)}{(1+r^2)} \ge 2, \quad r \in (0,1).$$

Using these estimates, we find from equation (7.12) and (7.19) that

$$(1-r)\sum_{n=0}^{\infty}\|A_d^n x\|^2 r^{2n} \le \frac{M^2+M_1^2+\frac{1}{4}}{2}\|x\|^2, \qquad r \in (0,1), \quad x \in X.$$

Since the norm of the adjoint equals the norm of the operator, we have that (7.18) also holds for A^* and A^{-*}. Thus, using analogous consideration for the functional $G_{A^*}(x;\lambda)$, we have

$$(1-r)\sum_{n=0}^{\infty}\|(A^*)_d^n x\|^2 r^{2n} \le \frac{M^2+M_1^2+\frac{1}{4}}{2}\|x\|^2, \qquad r \in (0,1), \quad x \in X.$$

Since $(A^*)_d = (A_d)^*$, we conclude from Lemma 2.23 that the power sequence $(A_d^n)_{n\in\mathbb{N}}$ is bounded and that equation (7.13) holds. □

As a corollary we state, that if a bounded operator in the Hilbert space is the generator of a bounded semigroup, then the powers of its cogenerator are power bounded, see also [21], [20], [2].

Corollary 7.7 *Let $A \in \mathcal{L}(X)$ be the generator of a bounded semigroup, then the power sequence of its cogenerator $(A_d^n)_{n\in\mathbb{N}}$ is bounded.*

Proof: The proof follows the same line as the proof of part 2 of Theorem 7.6. Since A generates a bounded semigroup we have by the Hille-Yosida Theorem that there exists a constant $M \ge 1$, such that

$$\|R^n(A,\lambda)\| \le M\lambda^{-n}, \qquad \lambda > 0, \quad n \in \mathbb{N}.$$

This means that for $\lambda > 1$ we can take the following sum:

$$\sum_{n=0}^{\infty} \left\| \left(\sqrt{\lambda^2 - 1}\right)^n R^n(A, \lambda) \right\|^2 \leq \sum_{n=0}^{\infty} M^2 \left(\frac{\lambda^2 - 1}{\lambda^2}\right)^n = M^2 \lambda^2. \tag{7.20}$$

Using Lemma 2.14, Theorem 2.15 and Remark 2.17 we know that there exists an unique solution for Lyapunov equation (2.45).
Since A generates a bounded semigroup and $A \in \mathcal{L}(X)$ we have that

$$\|(\lambda I - A)^{-1}(I - \lambda A)\| \leq M_3, \qquad \lambda > 1.$$

From Lemma 2.27 we know that equation (2.46) has a solution as well. Using Lemma 2.14 and Theorem 2.15 we have the following estimate:

$$\sum_{n=0}^{\infty} \left\| \left(\sqrt{\lambda^2 - 1}\right)^n R^n(A^{-1}, \lambda) \right\|^2 \leq M_4. \tag{7.21}$$

Combining equation (7.20) and equation (7.21) we have that equation (7.19) holds. From here the proof is the same as the proof of part 2 of Theorem 7.6. □

Corollary 7.8 *Let A generate a bounded analytic semigroup, then the power sequence of its cogenerator $(A_d^n)_{n\in\mathbb{N}}$ is bounded.*

Proof: First we define

$$\Sigma_\alpha := \left\{ s \in \mathbb{C} \;\middle|\; |\arg(s)| < \frac{\pi}{2} + \alpha, s \neq 0 \right\}.$$

Since A generate a bounded analytic semigroup, we know from Lemma 1.10 that

$$\|(sI - A)^{-1}\| \leq \frac{m}{|s|}, \qquad \text{for } s \in \Sigma_\alpha. \tag{7.22}$$

Now we show that for A^{-1} this equation holds as well. For $\frac{1}{\sigma} \in \Sigma_\alpha$ and using the triangle inequality, the following holds:

$$\begin{aligned}
\|(\sigma I - A^{-1})^{-1}\| &= \left\| -\frac{1}{\sigma} A \left(\frac{1}{\sigma} I - A\right)^{-1} \right\| \\
&= \left\| \left[\frac{1}{\sigma}\left(\frac{1}{\sigma} I - A\right) - \frac{1}{\sigma^2} I\right] \left(\frac{1}{\sigma} I - A\right)^{-1} \right\| \\
&\leq \left\| \frac{1}{\sigma}\left(\frac{1}{\sigma} I - A\right)\left(\frac{1}{\sigma} I - A\right)^{-1} \right\| + \left\| \frac{1}{\sigma^2} I \left(\frac{1}{\sigma} I - A\right)^{-1} \right\| \\
&\leq \frac{1}{|\sigma|} + \frac{1}{|\sigma|^2} m|\sigma| = \frac{m+1}{|\sigma|},
\end{aligned}$$

where we used equation (7.22) in the last estimate.
Since $\arg(s^{-1}) = -\arg s$ we know that $s \in \Sigma_\alpha \iff s^{-1} \in \Sigma_\alpha$. Now Lemma 1.10 implies that A^{-1} generates a bounded analytic semigroup. From Theorem 7.6 we know that $(A_d^n)_{n\in\mathbb{N}}$ is bounded. □

In Theorem 7.6 we have shown that if A and A^{-1} both generate a bounded semigroup on the Hilbert space X, then the powers of the cogenerator A_d^n are bounded. In the next lemma we give some other sufficient conditions.

Lemma 7.9 *Let A be a closed, densely defined linear operator on X, and let its spectrum $\sigma(A)$ be located in the half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq 0\}$. If one of the following conditions holds,*

1. *There exists an $s \in \mathbb{R}$ such that, $is \in \rho(A)$, and $-R(A, is)$ generates a bounded C_0-semigroup.*

2. *There exists a non-zero $s \in \mathbb{R}$, such that the operators $(A - isI)^{-1}$ and $(A + is^{-1}I)^{-1}$ generate a bounded C_0-semigroup.*

Then the powers sequence of the cogenerator $(A_d^n)_{n\in\mathbb{N}}$ is bounded.

Proof: Assume that condition 1. holds. Recall equation (1.19),

$$A_s = (-isA + I)(A - isI)^{-1}, \qquad D(A_s) = \operatorname{ran}(A - isI), \tag{7.23}$$

where we defined the operator A_s generates a bounded C_0-semigroup. Then, from condition 1. and Remark 1.17 we conclude that the bounded operator $A_s \in \mathcal{G}(X)$. Consequently, by Lemma 1.6 the power sequence of the cogenerator $(A_s)_d$ of the operator A_s is bounded. By Lemma 1.16 we have the equality

$$(A_s)_d = \alpha(s)A_d, \qquad |\alpha(s)| = 1$$

and thus the operator sequence $(A_d^n)_{n\in\mathbb{N}}$ is bounded as well.
Assume that condition 2. holds. From this condition we can define operators A_s, $A_{-s^{-1}}$, see equation (7.23). Furthermore, we have that$A_s^{-1} = A_{-s^{-1}}$. Moreover, from condition 2. and Remark 1.17 it follows that the operators A_s and $A_{-s^{-1}}$ generate a bounded C_0-semigroup. Applying Theorem 7.6 we conclude that the power sequence of the cogenerator $(A_s)_d$ is bounded. Using Lemma 1.16 we conclude that $(A_d^n)_{n\in\mathbb{N}}$ is bounded □

We remark that condition 2. is not stronger than condition 1. since in condition 1. it is assumed that $(A - isI)^{-1}$ is a bounded operator, whereas in condition 2. this operator is assumed to be an infinitesimal generator.
The following theorem extends the second statement in Theorem 7.6.

Theorem 7.10 *Let X be a Hilbert space, and let A generate a bounded C_0-semigroup on X. Assume further that the inverse of A exists as a closed, densely defined, linear operator. Then the following statements are equivalent:*

1. *For all $\varepsilon > 0$ the operator $-R(A,\varepsilon)$ generates a bounded C_0-semigroup and there exists constant $M_1 \geq 1$, which does not depend on ε, such that*
$$\|e^{-R(A,\varepsilon)t}\| \leq M_1, \qquad t \geq 0.$$
2. *The operator A^{-1} generates a bounded C_0-semigroup;*
3. *For all $\delta > 0$ the powers sequence of cogenerator $(\delta A)_d$ of the operator δA is bounded and there exists a constant $C \geq 1$ which does not depend on δ, such that*
$$\|(\delta A)_d^n\| \leq C, \qquad n = 0, 1, 2, \dots . \tag{7.24}$$

Proof: Implication 1. $\Rightarrow$ 2. follows from the second Trotter-Kato approximation theorem, see [16, Theorem III.4.9]. Moreover,

$$\|e^{A^{-1}t}\| \leq M_1, \qquad t \geq 0,$$

where M_1 is constant from condition 1.
2. $\Rightarrow$ 3. If A generates a bounded semigroup, so does δA. Furthermore,

$$\sup_{t\geq 0} \|e^{\delta A t}\| = \sup_{t \geq 0} \|e^{At}\| = M.$$

Similarly, we have that

$$\sup_{t\geq 0} \|e^{(\delta A)^{-1} t}\| = \sup_{t \geq 0} \|e^{A^{-1}t}\| = M_1.$$

By part 2 of Theorem 7.6, we have that the power sequence of $(\delta A)_d$ is bounded, with a bound independent of δ. Thus we have proved (7.24) with $C = \frac{e}{2}(M^2 + M_1^2 + \frac{1}{4})$.
3. $\Rightarrow$ 1. Using equation (7.24), we see that for any $\delta > 0$ we have the inequality

$$\|e^{(\delta A)_d t}\| \leq C e^t, \qquad t \geq 0. \tag{7.25}$$

Let $\varepsilon > 0$, by equation (1.17), we have that

$$e^{-2\varepsilon t R(A,\varepsilon)} = e^{2t(\varepsilon^{-1}A - I)^{-1}} = e^{-t} e^{(\varepsilon^{-1}A)_d t}.$$

Choosing $\delta = \varepsilon^{-1}$ and combining this with (7.25), we find that $\|e^{tR(A,\epsilon)}\| \leq C$, $t \geq 0$. With this we have obtained condition 1. □

7.3 Conclusions

In the previous chapters we looked at the relation of the growth of the semigroup and the growth of the powers of the cogenerator. In this chapter we included the growth of the semigroup generated by the inverse operator.
In Section 7.1 we assumed A to generate an exponentially stable semigroup on a Banach space. Under these conditions the growth of the powers of the cogenerator and the growth of the semigroup generated by the inverse are similar. In Theorem 7.1 and Theorem 7.5 we showed that a growth bound on one implies the same growth bound on the other. As corollaries we stated that the growth of semigroup $\left(e^{A^{-1}t}\right)_{t\geq 0}$ is bounded by $t^{\frac{1}{4}}$ on a Banach space, and by $\ln(t+1)$ on a Hilbert space.
In Section 7.2 we assumed A to generate a bounded semigroup on a Hilbert space. We focus on conditions for which the powers of the cogenerator are bounded. In Theorem 7.6 we showed that, for this, it is sufficient for A^{-1} to generated a bounded semigroup. As a corollary we state that if a bounded operator, i.e. $A \in \mathcal{L}(X)$, generates a bounded semigroup, or A generates a bounded analytic semigroup, then the power sequence of the cogenerator is bounded. In Theorem 7.10 we showed that the condition that the inverse generates a bounded semigroup is equivalent to the condition that the resolvent operator $R(A,\varepsilon)$ generates a bounded semigroup for all $\varepsilon > 0$. And that this is equivalent with the boundedness of the power sequence of the cogenerator of δA, for all $\delta > 0$.

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Summary

The relation between continuous time systems and discrete time systems is the main topic of this research. A continuous time system can be transformed into a discrete time system using the Cayley transform. In this process, the solutions of the continuous time system, given by a semigroup, are mapped to the solutions of the discrete time system, given by a power sequence. To be more precise, the generator of the semigroup is mapped to a difference operator, the cogenerator. Stability analysis plays a central role in the study of the relation between continuous and discrete time systems. Do stable continuous time systems correspond to stable discrete time systems? This is the main question of this dissertation.

For many stable continuous time systems the corresponding discrete time systems are stable as well. In Banach spaces however, several examples are known of stable semigroups where the corresponding cogenerators have unstable power sequences. In Hilbert spaces no such examples are known. It remains an open problem whether for every stable continuous time system the corresponding discrete time system is stable as well.

This dissertation addresses the main question in three ways.

First, a growth bound for the cogenerator is provided. This bound holds for exponentially stable semigroups in Hilbert spaces. Using Lyapunov equations it is shown that for such semigroups the power sequence of the corresponding cogenerator cannot grow faster than $\ln(n)$.

Second, we extend the class of stable continuous time systems for which the corresponding discrete time systems are stable as well. For this, the notion of Bergman distance is introduced. The Bergman distance defines a metric for semigroups and a metric for power sequences. In other words, the Bergman distance is a distance between two semigroups or two power sequences. If the Bergman distance is finite, the two semigroups have the same stability behaviour. This holds for two power sequences as well. Furthermore, the Bergman distance is preserved by the Cayley transform. This means that the Bergman distance of two semigroups is equal to the Bergman distance

of the power sequences of their corresponding cogenerators. Combined with the stability properties of the Bergman distance, this enables us to identify new stable continuous time systems for which the corresponding discrete time systems are stable as well.
Third, the inverse of the generator is taken into account in the stability analysis. For stable semigroups the growth of the power sequence of the cogenerator is related to the growth of the semigroup generated by the inverse of the generator. For exponentially stable semigroups on Banach spaces similarity is shown between the growth of the semigroup of the inverse and the growth of the power sequence of the cogenerator. Bounded semigroups on Hilbert spaces are examined as well. It is shown that if the semigroup generated by the inverse is bounded, the growth of the cogenerator is bounded as well.

Samenvatting

Het centrale thema van dit onderzoek is de relatie tussen systemen waarvoor tijd een continue grootheid is en systemen waarvoor tijd een discrete grootheid is. Continue systemen kunnen via de Cayleytransformatie worden getransformeerd naar discrete systemen. In dit proces worden de oplossingen van het continue systeem, die worden beschreven door een halfgroep, gekoppeld aan de oplossingen van het discrete systeem, die worden beschreven door een machtreeks. Om precies te zijn wordt de generator van de halfgroep afgebeeld op een differentieoperator, de cogenerator. Binnen het onderzoek naar de relatie tussen continue en discrete systemen speelt stabiliteitsanalyse een belangrijk rol.

In dit proefschrift staat de volgende vraag centraal: worden stabiele continue systemen getransformeerd naar stabiele discrete systemen?

Voor veel stabiele continue systemen geldt dat het bijbehorende discrete systeem ook stabiel is. In Banachruimtes zijn echter een aantal voorbeelden bekend van stabiele halfgroepen waarbij de machtreeks van de corresponderende cogenerator niet stabiel is. In Hilbertruimtes zijn geen dergelijke voorbeelden bekend. Het blijft een open probleem of in Hilbertruimtes voor alle stabiele continue systemen de bijbehorende discrete systemen stabiel zijn.

In dit proefschrift wordt de centrale onderzoeksvraag op drie manieren behandeld.

In de eerste plaats wordt een grens gegeven voor de groei van de machtreeks van de cogenerator. Deze grens geldt voor exponentieel stabiele halfgroepen in Hilbertruimtes. Door gebruik te maken van Lyapunovvergelijkingen wordt bewezen dat voor dergelijke halfgroepen de machtreeks van de bijbehorende cogenerator niet sneller kan groeien dan $\ln(n)$.

In de tweede plaats wordt een uitbreiding gegeven op de klasse van stabiele continue systemen waarvoor het bijbehorende discrete systeem eveneens stabiel is. Hiervoor wordt de Bergmanafstand geïntroduceerd. De Bergmanafstand definieert een metriek voor halfgroepen en een metriek voor

machtreeksen. Dit betekent dat de Bergmanafstand een afstand is tussen twee halfgroepen of twee machtreeksen. Indien de Bergmanafstand eindig is, hebben de twee halfgroepen hetzelfde stabiliteitsgedrag. Bovendien wordt de Bergmanafstand behouden onder de Cayleytransformatie. Dit betekent dat twee halfgroepen dezelfde Bergmanafstand hebben als de machtreeksen van de bijbehorende cogeneratoren. Gecombineerd met de stabiliteitseigenschappen van de Bergmanafstand biedt dit de mogelijkheid nieuwe stabiele continue systemen te identificeren waarvoor het bijbehorende discrete systeem eveneens stabiel is.
In de derde plaats wordt de inverse van de generator bij de stabiliteitsanalyse betrokken. Voor stabiele halfgroepen is de groei van de machtreeks van de cogenerator gerelateerd aan de groei van de halfgroep gegenereerd door de inverse van de generator. In Banachruimtes wordt voor exponentieel stabiele halfgroepen bewezen dat de groei van de halfgroep van de inverse vergelijkbaar is met de groei van de machtreeks van de cogenerator. Tevens wordt naar begrensde halfgroepen in Hilbertruimtes gekeken. Er wordt bewezen dat als de inverse van de generator een begrensde halfgroep genereert, de groei van de machtreeks van de cogenerator ook begrensd is.

About the author

Niels Besseling was born on 2 September 1979 in Hoorn, the Netherlands. He received his vwo diploma from the S.G. Oscar Romero school in Hoorn and participated in the International Mathematical Olympiad in Mar del Plata, Argentina.
Afterwards, Niels moved to Enschede to study Applied Mathematics at the University of Twente. As part of this study he obtained a minor in Business Administration and performed a practical training at the Institute Laue-Langevin in Grenoble, France. His Master's thesis, written under the supervision of Prof. Ruud Martini, was on the subject of Hopper's equation.
In 2007 Niels started as a Ph.D. student at the Stochastic System and Signal Theory group and the Mathematical Systems & Control Theory group at the University of Twente under the supervision of Prof. Hans Zwart. The results of this research are written down in this dissertation. As a Ph.D. student he participated in the research school DISC (Dutch Institute of Systems and Control).
During his time as a Ph.D. student, Niels was a member of the Emergency Response Team at the University of Twente. He played football for v.v. Drienerlo, the local student football club and likes cycling, preferably in the French Alpes. In 2010 he started working as a software engineer at OVSoftware.
Niels lives in Enschede with his wife Susanne and his son Daan.